NOTE: This examination is closed book. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of a problem to solve the other part, even if you are unable to solve the part being used. A complete and clearly written solution of a problem will get a more favorable review than a partial solution.

You must start solution of each problem on the given page. Be sure to put the number assigned to you on the right corner top of every page of your solution.
1. Let $X_1, \ldots, X_n$ be i.i.d. from a Pareto distribution with the parameters $\lambda > 2$ and $x_0 > 0$, i.e. from a density

$$f(x; \lambda, x_0) = \frac{\lambda x_0^\lambda}{x^{\lambda+1}}, x \geq x_0,$$

and 0 elsewhere.

(a) Suppose $x_0$ is known. Find a complete and sufficient statistics for $\lambda$ based on $X_1, \ldots, X_n$.

(b) Suppose $x_0$ is known. Let $g(\lambda) = \lambda^{-2}$. Formulate the two methods of finding UMVUE and use one of them to find it for $g(\lambda)$.

(c) Suppose $x_0$ is known. Find the MLE of $g(\lambda)$. Using delta method find its asymptotical variance, $n > 1$.

(d) Suppose $x_0$ is known. Let $\lambda$ have a distribution with the density $f(\lambda) = \theta e^{-\theta \lambda}$, $\lambda > 0$. Determine the Bayes estimator of $\lambda$ under the loss $\lambda(\delta(X_1, \ldots, X_n) - \lambda)^2$. Hint: first prove that the estimator has the form $E(\lambda^2 | X)/E(\lambda | X)$.

(e) Suppose $x_0$ is known. Calculate the risk of the Bayes estimator in (d). Find conditions on $\theta$ that would make the estimator in (d) minimax. Are they satisfied, $n > 1$? Hint: first prove that the posterior risk has the form $E(\lambda^3 | X) - (E(\lambda^2 | X))^2/E(\lambda | X)$.

(f) Suppose $x_0$ is known. Check whether the Bayes estimator of $\lambda$ in (d) dominates the MLE under the loss $\lambda(\delta(X_1, \ldots, X_n) - \lambda)^2$.

(g) Suppose $x_0$ is known. Derive a UMP unbiased test of size $\alpha \in (0, 1)$ for testing $H : \lambda \in [3, 4]$ vs $K : \lambda \in (2, 3) \cup (4, +\infty)$ in full detail.

(h) Find the $m$-th moment of $X_1$, $m \geq 1$.

(i) Suppose $\lambda$ is known. Show that $\min(X_1, \ldots, X_n) n^{-3/\lambda}$ is the MRE estimator of $x_0$ under the loss $(\delta(X_1, \ldots, X_n)/x_0 - 1)^2$, $n > 1$.

(j) Is the estimator in (i) consistent in probability for $x_0$?

(k) Derive a UMP test of size $\alpha \in (0, 1)$ for testing $H : x_0 = 5$ vs $K : x_0 > 5$ in full detail.

(l) Derive the power of the test in (k).

2. Consider the linear regression model $y = X\beta + \varepsilon$ where $X$ is an $n \times p$ full rank design matrix. Suppose we partition the model as $y = X_1\beta_1 + X_2\beta_2 + \varepsilon$ where $X_1$ is $n \times p_1$ and $X_2$ is $n \times p_2$. Moreover, $\varepsilon$ is normally distributed with mean 0 and variance $\sigma^2 I_n$.

(a) Show that the least squares estimate of $\beta_2$ can be written as $\hat{\beta}_2 = C'y$ where $C = X_1C_{12} + X_2C_{22}$, $C_{12}$ and $C_{22}$ are matrices of orders $p_1 \times p_2$ and $p_2 \times p_2$, respectively, that depend on $X_1$ and $X_2$, and $C_{22}$ is positive definite.

(b) Find a linear combination of $y$ (i.e., $Qy$) so that $Qy$ follows the Gauss Markov model with only coefficients $\beta_2$ in the model.

(c) Show that $[I_n - X_1(X_1'X_1)^{-1}X_1]'C$ is of rank $p_2$, where $C$ is defined in part (a).

(d) Let $SS_{E_1}$ be the residual sum of squares for the regression of $y$ on $X_1$ alone. Using part (c) to show that $SS_{E_1}$ and $\hat{\beta}_2$ are not independent.

(e) Construct a $1 - \alpha$ level confidence region for $\beta_2$.

(f) Assume $\beta_2 = (\beta_{21}, \ldots, \beta_{2p_2})^T$. Using Scheffe’s method to find $1 - \alpha$ simultaneous confidence intervals for $\beta_{2j}$ for $j = 1, \ldots, p_2$. 


3. Consider the case of a linear model \( E[y] = X\beta \) and \( \text{var}(y) = V \). Assume that \( X \) is of full column rank, and use the notation of \( \hat{\beta}_W^{-1} \) to denote the weighted least squares estimator \( (X'WX)^{-1}X'Wy \). Thus \( \hat{\beta}_I \) denote the ordinary least squares estimator and \( \hat{\beta}_V \) denotes the MLE with a known variance-covariance matrix \( V = \text{var}(y) \), i.e., \( \hat{\beta}_V = (X'V^{-1}X)^{-1}X'V^{-1}y \).

(a) Show that \( \hat{\beta}_I \) is unbiased no matter what the value of \( V \).

(b) Calculate the variance of \( \hat{\beta}_I \).

(c) Show that \( \hat{\beta}_V \) is unbiased.

(d) Calculate the variance of \( \hat{\beta}_V \).

(e) Is \( \hat{\beta}_V \) smaller than \( \hat{\beta}_I \)? Then why?

(f) Consider the simple situation where the mean of all the observations is \( \mu \) and the data come in \( m \) equicorrelated clusters with correlation \( \rho > 0 \). Further assume that half the clusters are of size \( n \) and the other half are of size \( \lambda n \). We thus have
\[
y \sim (1\mu, V),
\]
where
\[
V = \begin{bmatrix}
I_{m/2} \otimes V_{0,n} & 0 \\
0 & I_{m/2} \otimes V_{0,\lambda n}
\end{bmatrix}
\]
and \( V_{0,n} = \sigma^2[(1 - \rho)I_n + \rho J_n] \).

Derive the relative efficiency of \( \hat{\beta}_I \) and \( \hat{\beta}_V \).