STT 872, 867-868 Spring Preliminary Examination
Wednesday, January 6, 2016
12:30 - 5:30 pm

NOTE: This examination is closed book. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of a problem to solve the other part, even if you are unable to solve the part being used. A complete and clearly written solution of a problem will get a more favorable review than a partial solution.

You must start solution of each problem on the given page. Be sure to put the number assigned to you on the right corner top of every page of your solution.

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1. Let $X_1, ..., X_n$ be i.i.d. from the Poisson distribution with the parameter $\lambda > 0$.

(a) Find a complete and sufficient statistics for λ based on $X_1, ..., X_n$.

(b) Let $g(\lambda) = \lambda^3$. Formulate the two methods of finding UMVUE and use one of them to find it for $g(\lambda)$.

(c) Find the MLE of $g(\lambda)$. Using delta method find its asymptotical variance, $n > 1$.

(d) Let λ have a Gamma distribution with the density $\frac{x^{(a-1)}e^{-x/b}}{\Gamma(a)b^a}$, $x > 0$. Determine the Bayes estimator of λ under the loss $\lambda(\delta(X_1, ..., X_n) - \lambda)^2$.

(e) Calculate the risk of the Bayes estimator in (d). Find conditions on (a, b) that would make the estimator in (d) minimax. Are they satisfied, $n > 1$?

(f) Check whether the Bayes estimator of λ in (d) is inadmissible under the loss $\lambda(\delta(X_1, ..., X_n) - \lambda)^2$. Hint: compare with MLE.

(g) Derive a UMP unbiased test of size $\alpha \in (0, 1)$ for testing $H : \lambda \in [1, 2]$ vs $K : \lambda \in (0, 1) \cup (2, +\infty)$ in full detail.

2. Let $X_1, ..., X_n$ be i.i.d. from the density $\frac{I(0<x<\lambda)}{2\sqrt{\lambda x}}, \lambda > 0$.

(a) Find the m-th moment of X_1, $m \geq 1$.

(b) Construct the MRE estimator of λ under the loss $(\delta(X_1, ..., X_n)/\lambda - 1)^2, n > 1$.

(c) Is the estimator in (b) consistent in probability for λ?

(d) Derive a UMP test of size $\alpha \in (0, 1)$ for testing $H : \lambda = 5$ vs $K : 0 < \lambda < 5$ in full detail.

(e) Derive the power of the test in (d).

3. Consider the following model

$$Y_{ij} = \mu + \alpha_i + \beta_j + e_{ij} \text{ where } e_{ij} \overset{iid}{\sim} N(0, \sigma^2) \tag{1}$$

for $i = 1, 2, 3$ and $j = 1, 2, 3, 4,$. The model is analyzed using the following data set

		β_j			
		1	2	3	4
α_i	1	12	-	22	30
	2	-	-	4	11
	3	20	15	12	-

where - represents the empty cell. For example, the cell corresponding to (α_2, β_1) is empty. In another word, Y_{21} is missing.

(a) Specify the design matrix X of the linear model (1) corresponding to the above data set. What is the number of basic linearly independent estimable linear functions for this data set?

(b) Is $\mu_{12} = \mu + \alpha_1 + \beta_2$ estimable? Why or why not?

(c) Show that $\frac{1}{4}\sum_{j=1}^{4}(\mu + \alpha_i + \beta_j)$ is estimable for $i = 1, 2, 3$. Find the corresponding best linear unbiased estimators.

(d) Let $\phi_j = \frac{1}{3}\sum_{i=1}^{3}(\mu+\alpha_i+\beta_j)$. Consider the hypothesis $H_0 : \phi_1 = \phi_2 = \phi_3 = \phi_4$ vs. $H_1 : \phi_k \neq \phi_l$ for some $k \neq l \in \{1, 2, 3, 4\}$. Is this hypothesis testable? Why or why not? If testable, what is the corresponding test statistic and the α-level rejection region?

(e) Consider the pairwise difference among ϕ_k and ϕ_l for $k \neq l \in \{1, 2, 3, 4\}$. Construct individual and Scheffe's simultaneous confidence intervals for $\phi_k - \phi_l$ with $k \neq l \in \{1, 2, 3, 4\}$.

4. Five batches of raw material were randomly selected. Several random samples were randomly taken from each batch and the purity of the material was determined from each sample. The following data were obtained:

Batches				
1	2	3	4	5
10.93	14.72	9.97	21.51	18.45
12.71	15.91	10.78	20.75	17.25
11.35	17.10		19.69	16.95
13.50				

The corresponding model is the one-way model given below

$$Y_{ij} = \mu + \alpha_i + \epsilon_{i(j)}$$

where α_i is the random batch effect and $\epsilon_{i(j)}$ is the measurement error. Assume that α_i are iid $N(0, \sigma_\alpha^2)$ and $\epsilon_{i(j)}$ are iid $N(0, \sigma^2)$. Moreover, assume that α_i is independent of $\epsilon_{i(j)}$.

(a) Please provide the one-way ANOVA table for the above model using the given data set. Based on the one-way ANOVA table, construct unbiased estimators for σ_α^2 and σ^2.

For the following three questions, move the data point 13.50 from Batch 1 to Batch 3.

(b) Construct a test statistic for testing $H_0 : \sigma_\alpha^2 = 0$ vs $H_1 : \sigma_\alpha^2 \neq 0$.

(c) What is the distribution of the statistic in part (a) under H_0 and the alternative H_1 in part (b).

(d) Show how you can compute the power of the test in part (b) for an α-level of significant test and a specific known value of σ_α^2/σ^2.