STT 872, 867-868 Fall Preliminary Examination
Wednesday, August 23, 2017
12:30 - 5:30 pm

NOTE: This examination is closed book. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of a problem to solve the other part, even if you are unable to solve the part being used. A complete and clearly written solution of a problem will get a more favorable review than a partial solution.

You must start solution of each problem on the given page. Be sure to put the number assigned to you on the right corner top of every page of your solution.

1. Let $X_1, X_2, \ldots, X_n$ be i.i.d. Bernoulli random variables with a parameter $p \in (0,1)$. Formulate the two methods of finding UMVUE and use one of them to find it for $p^k, k < n$.

2. Let $X_1, \ldots, X_n$ be i.i.d. absolutely continuous random variables with the common density

$$f_\theta(x) = \begin{cases} x\theta^2 e^{-\theta x}, & x > 0; \\ 0, & x \leq 0. \end{cases}$$

Let $p = g(\theta) = (1+\theta)e^{-\theta} = P_\theta(X_i > 1)$. Two natural estimators for p are

$$\tilde{p}_n = \#\{i : X_i > 1\}/n \qquad \text{and} \qquad \hat{p}_n = g(\hat{\theta}_n),$$

where $\hat{\theta}_n$ is the maximum likelihood estimator of θ.

(a) Find the limiting distribution of $\sqrt{n}(\tilde{p}_n - p)$.

(b) Find the limiting distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$,

(c) Derive the asymptotic relative efficiency of $\tilde{p}_n$ with respect to $\hat{p}_n$.

3. A random vector X takes values $(1,0,0), (0,1,0), (0,0,1)$ with the corresponding probabilities $\theta_1, (1-\theta_1)\theta_2, (1\theta_1)(1-\theta_2)$, where $(\theta_1, \theta_2) \in (0,1) \times (0,1)$. Let $X_1, X_2, \ldots, X_n$ be i.i.d. observations of X.

(a) Find the MLE of (θ_1, θ_2) .

(b) Find the Bayes estimators of (θ_1, θ_2) when $\theta_i \sim Beta(\alpha_i, \beta_i), i = 1, 2$ and the loss function is $L((\theta_1, \theta_2), (a, b)) = (\theta_1 - a)^2 + (\theta_2 - b)^2$.

(c) Show that the MLE is admissible.

4. Suppose that the parameter space in a decision problem is $\Theta = \Theta_1 \times \Theta_2$. If δ_0 is a decision rule such that, for each fixed $\theta_i \in \Theta_1$, δ_0 is minimax for the decision problem with parameter space $\{(\theta_0, \theta_2) : \theta_2 \in \Theta_2\}$, show that δ_0 is minimax when the parameter space is $\Theta_1 \times \Theta_2$.

5. Let U be uniformly distributed on $(0,1)$. Suppose we observe a single observation X, and that under H_0, $X \sim U$, and under H_1, $X \sim U^2$. Find the best test $\varphi(X)$ of H_0 versus H_1 with level $\alpha = 1/2$. What is the power of this test when H_1 is correct?

6. Let $X_1, X_2, \ldots, X_n$ be i.i.d. $U(\theta, \theta+2) : \theta = 0, 1, 2, \ldots$. Let $d(X_1, X_2, \ldots, X_n)$ be the nearest integer to $\max X_i$. Show d is not admissible under 0-1 loss by constructing a decision rule that is better than d.

7. Let $X_1, X_2, \ldots, X_n$ be i.i.d. from an exponential distribution with parameter $\tau > 0$, i.e. $f(x;\tau) = \frac{1}{\tau}e^{-x/\tau}, x > 0$. Derive a UMP unbiased test of size $\alpha \in (0,1)$ for testing $H : \tau \in [4,5]$ vs $K : \tau \in (0,4) \cup (5, +\infty)$ in full detail.

8. In the following questions, please use the following notations to answer questions: $r_\alpha = \sigma_\alpha^2/\sigma^2$; $\mathbf{1}_q$ represents a column vector of 1's with length q. For instance, $\mathbf{1}_5 = (1,1,1,1,1)^T$. Let $\underset{\sim}{y}_i = (Y_{i1}, \cdots, Y_{i5})^T$ and $\underset{\sim}{\varepsilon}_i = (\varepsilon_{i1}, \cdots, \varepsilon_{i5})^T$. Moreover, denote

$$\bar{Y}_{i\cdot} = \frac{1}{5}\sum_{j=1}^{5} Y_{ij} \quad \text{and} \quad \bar{Y}_{\cdot\cdot} = \frac{1}{15}\sum_{i=1}^{3}\sum_{j=1}^{5} Y_{ij}.$$

Consider a one-way ANOVA random effects model:

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \qquad i = 1, \cdots, 3; j = 1, \cdots, 5$$

where α_i's are iid $N(0, \sigma_\alpha^2)$ and ε_{ij}'s are iid $N(0, \sigma^2)$. Moreover, assume that α_i and ε_{ij} are independent.

(a) Write the one-way ANOVA random effect model as a general mixed effects model of form $\underset{\sim}{y} = \mathbf{X}\underset{\sim}{\beta} + \mathbf{Z}\underset{\sim}{b} + \underset{\sim}{\varepsilon}$, where $\underset{\sim}{\beta}$ are fixed effects and $\underset{\sim}{b}$ are random effects. Please specify $\mathbf{X}, \mathbf{Z}, \underset{\sim}{\beta}$ and $\underset{\sim}{b}$.

(b) Apply the mixed-effect equations to find the best linear unbiased predictor (BLUP) for $\mu + \alpha_2$. Write the BLUP as a linear combination of $\bar{Y}_{2\cdot}$ and $\bar{Y}_{\cdot\cdot}$.

Consider the above one-way ANOVA random effects model. Assume that the BLUP for $\mu + \alpha_2$ is

$$\widehat{\mu + \alpha_2} = a\bar{Y}_{2\cdot} + (1-a)\bar{Y}_{\cdot\cdot}$$

where a is a constant.

(c) Find the mean square error (MSE) for the BLUP of $\mu + \alpha_2$. For what value of a, the MSE in part (a) is minimized?

(d) Construct a 95% prediction interval for $\mu + \alpha_2$ using the BLUP.

Consider the above one-way ANOVA model as a fixed effects model. Namely, treat α_i's as fixed effects.

(e) Give the test statistic and α-level rejection region for testing $H_0 : \alpha_1 = \alpha_2 = \alpha_3$ vs H_1 : one pair of α_i and α_j $(i \neq j)$ are not equal.

(f) Use Scheffé's method to construct 95% simultaneous confidence intervals for $\alpha_i - \alpha_j$ $(i \neq j \in \{1, 2, 3\})$.