NOTE: This examination is closed book. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of a problem to solve the other part, even if you are unable to solve the part being used. A complete and clearly written solution of a problem will get a more favorable review than a partial solution.

You must start solution of each problem on the given page. Be sure to put the number assigned to you on the right corner top of every page of your solution.
1. Let \( X_1, X_2, \ldots, X_n \) be i.i.d. Bernoulli random variables with a parameter \( p \in (0, 1) \). Formulate the two methods of finding UMVUE and use one of them to find it for \( p^k, k < n \).

2. Let \( X_1, \ldots, X_n \) be i.i.d. absolutely continuous random variables with the common density

\[
f_\theta(x) = \begin{cases} 
  x\theta^2 e^{-\theta x}, & x > 0; \\
  0, & x \leq 0.
\end{cases}
\]

Let \( p = g(\theta) = (1 + \theta)e^{-\theta} = P_\theta(X_i > 1) \). Two natural estimators for \( p \) are

\[
\hat{p}_n = \frac{\#\{i : X_i > 1\}}{n} \quad \text{and} \quad \hat{p}_n = g(\hat{\theta}_n),
\]

where \( \hat{\theta}_n \) is the maximum likelihood estimator of \( \theta \).

(a) Find the limiting distribution of \( \sqrt{n}(\hat{p}_n - p) \).

(b) Find the limiting distribution of \( \sqrt{n}(\hat{\theta}_n - \theta) \).

(c) Derive the asymptotic relative efficiency of \( \hat{p}_n \) with respect to \( \hat{p}_n \).

3. A random vector \( X \) takes values \((1, 0, 0), (0, 1, 0), (0, 0, 1)\) with the corresponding probabilities \(\theta_1, (1-\theta_1)\theta_2, (\theta_1)(1-\theta_2)\), where \((\theta_1, \theta_2) \in (0, 1) \times (0, 1)\). Let \( X_1, X_2, \ldots, X_n \) be i.i.d. observations of \( X \).

(a) Find the MLE of \((\theta_1, \theta_2)\).

(b) Find the Bayes estimators of \((\theta_1, \theta_2)\) when \(\theta_i \sim Beta(\alpha_i, \beta_i), i = 1, 2\) and the loss function is

\[L((\theta_1, \theta_2), (a, b)) = (\theta_1 - a)^2 + (\theta_2 - b)^2.\]

(c) Show that the MLE is admissible.

4. Suppose that the parameter space in a decision problem is \( \Theta = \Theta_1 \times \Theta_2 \). If \( \delta_0 \) is a decision rule such that, for each fixed \( \theta_i \in \Theta_1 \), \( \delta_0 \) is minimax for the decision problem with parameter space \( \{(\theta_0, \theta_2) : \theta_2 \in \Theta_2\} \), show that \( \delta_0 \) is minimax when the parameter space is \( \Theta_1 \times \Theta_2 \).

5. Let \( U \) be uniformly distributed on \((0, 1)\). Suppose we observe a single observation \( X \), and that under \( H_0, X \sim U \), and under \( H_1, X \sim U^2 \). Find the best test \( \varphi(X) \) of \( H_0 \) versus \( H_1 \) with level \( \alpha = 1/2 \). What is the power of this test when \( H_1 \) is correct?

6. Let \( X_1, X_2, \ldots, X_n \) be i.i.d. \( U(\theta, \theta + 2) : \theta = 0, 1, 2, \ldots \). Let \( d(X_1, X_2, \ldots, X_n) \) be the nearest integer to \( \max X_i \). Show \( d \) is not admissible under 0-1 loss by constructing a decision rule that is better than \( d \).

7. Let \( X_1, X_2, \ldots, X_n \) be i.i.d. from an exponential distribution with parameter \( \tau > 0 \), i.e.

\[f(x; \tau) = \frac{1}{\tau} e^{-x/\tau}, x > 0.\]

Derive a UMP unbiased test of size \( \alpha \in (0, 1) \) for testing \( H : \tau \in [4, 5] \) vs \( K : \tau \in (0, 4) \cup (5, +\infty) \) in full detail.
8. In the following questions, please use the following notations to answer questions: \( r_{\alpha} = \sigma_{\alpha}^2 / \sigma^2 \); \( \mathbf{1}_q \) represents a column vector of 1’s with length q. For instance, \( \mathbf{1}_5 = (1,1,1,1,1)^T \). Let \( \mathbf{y}_i = (Y_{i1}, \cdots, Y_{i5})^T \) and \( \mathbf{\varepsilon}_i = (\varepsilon_{i1}, \cdots, \varepsilon_{i5})^T \). Moreover, denote

\[
\bar{Y}_i = \frac{1}{5} \sum_{j=1}^{5} Y_{ij} \quad \text{and} \quad \bar{Y}_* = \frac{1}{15} \sum_{i=1}^{3} \sum_{j=1}^{5} Y_{ij}.
\]

Consider a one-way ANOVA random effects model:

\[
Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad i = 1, \cdots, 3; j = 1, \cdots, 5
\]

where \( \alpha_i \)'s are iid \( N(0, \sigma_{\alpha}^2) \) and \( \varepsilon_{ij} \)'s are iid \( N(0, \sigma^2) \). Moreover, assume that \( \alpha_i \) and \( \varepsilon_{ij} \) are independent.

(a) Write the one-way ANOVA random effect model as a general mixed effects model of form

\[
y = X\beta + Z\mathbf{b} + \mathbf{\varepsilon},
\]

where \( \beta \) are fixed effects and \( \mathbf{b} \) are random effects. Please specify \( X, Z, \beta \) and \( \mathbf{b} \).

(b) Apply the mixed-effect equations to find the best linear unbiased predictor (BLUP) for \( \mu + \alpha_2 \). Write the BLUP as a linear combination of \( \bar{Y}_2 \) and \( \bar{Y}_* \).

Consider the above one-way ANOVA random effects model. Assume that the BLUP for \( \mu + \alpha_2 \) is

\[
\hat{\mu} + \hat{\alpha}_2 = a\bar{Y}_2 + (1-a)\bar{Y}_*.
\]

where \( a \) is a constant.

(c) Find the mean square error (MSE) for the BLUP of \( \mu + \alpha_2 \). For what value of \( a \), the MSE in part (a) is minimized?

(d) Construct a 95% prediction interval for \( \mu + \alpha_2 \) using the BLUP.

Consider the above one-way ANOVA model as a fixed effects model. Namely, treat \( \alpha_i \)'s as fixed effects.

(e) Give the test statistic and \( \alpha \)-level rejection region for testing \( H_0 : \alpha_1 = \alpha_2 = \alpha_3 \) vs \( H_1 : \) one pair of \( \alpha_i \) and \( \alpha_j \) \((i \neq j)\) are not equal.

(f) Use Scheffé’s method to construct 95% simultaneous confidence intervals for \( \alpha_i - \alpha_j \) \((i \neq j \in \{1, 2, 3\})\).