STT 872, 867-868 Fall Preliminary Examination
Wednesday, August 24, 2016
12:30 - 5:30 pm

NOTE: This examination is closed book. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of a problem to solve the other part, even if you are unable to solve the part being used. A complete and clearly written solution of a problem will get a more favorable review than a partial solution.

You must start solution of each problem on the given page. Be sure to put the number assigned to you on the right corner top of every page of your solution.

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1. Let $X_1, ..., X_n$ be i.i.d. from the log normal distribution with the parameter $\lambda > 0$. Hint: the density is $f(x; \lambda) = \frac{\exp\{-0.5(\ln x)^2/\lambda^2\}}{x\lambda\sqrt{2\pi}}, x > 0$.

(a) Find a complete and sufficient statistics for λ based on $X_1, ..., X_n$.

(b) Let $g(\lambda) = \lambda^4$. Formulate the two methods of finding UMVUE of a parameter and use one of them to find it for $g(\lambda)$.

(c) Find the MLE of $g(\lambda)$. Using delta method find its asymptotic variance, $n > 1$.

(d) Construct the MRE estimator of $g(\lambda)$ under the loss $(\delta(X_1, ..., X_n)/\lambda^4 - 1)^2, n > 1$.

(e) Derive a UMP unbiased test of size $\alpha \in (0, 1)$ for testing $H : \lambda \geq 2$ vs $K : \lambda < 2$ in full detail.

(f) Derive the power function of the test in (f).

2. Let

$$f_j(x) := \frac{1}{2\beta_j^3} x^2 e^{-x/\beta_j}, \quad x > 0,\ \beta_j > 0,\ j = 1, 2.$$

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ represent two independent random samples from f_1 and f_2, respectively. Let $0 < \alpha < 1$. Derive the uniformly most powerful unbiased test of size α for testing $H : \beta_2 = \beta_1$ against the alternative $K : \beta_2 > \beta_1$.

3. Let θ have a possibly improper prior Lebesgue density g on $(0, \infty)$ (i.e., *g is nonnegative but $\int g(\theta)d\theta$ may be infinity*) and let X be a positive r.v. whose conditional Lebesgue density, given θ, is $f(\cdot|\theta)$. Assume $g,\ f$ to be such that

$$E(\theta^2|X = x) < \infty, \qquad \forall\ x > 0.$$

Consider the problem of estimating θ under the loss function

$$L(\theta,\ t) := (\theta - t)^2/t, \qquad t > 0.$$

(a) Show that the Bayes estimator of θ is $T(X) = \sqrt{E(\theta^2|X)}$.

(b) Compute this estimator in the case

$$g(\theta) = 1/\theta, \qquad f(x|\theta) := \frac{x^{a-1}e^{-x/\theta}}{\theta^a\Gamma(a)}, \qquad x > 0,\ \theta > 0,\ a > 2.$$

4. Let f_1 and f_2 denote densities of uniform distribution on $\{0, 1\}$ and $(0, 1)$, respectively. Let X denote an observation from a density f. Consider the problem of testing $H : f = f_1$ versus the alternative $K : f = f_2$. Give a test of size 0.0 and power 1.0 based on X for testing H versus K.

5. Consider the following model:

$$Y_{ij} = \mu + \alpha_i + w_i\gamma + \varepsilon_{ij} \qquad i = 1, \cdots, I; j = 1, \cdots, 5,$$

where w_i's are 1-dimensional observable covariates, γ is a 1-dimensional unknown coefficient, and ε_{ij}'s are independent and identically distributed (i.i.d.) $N(0, \sigma^2)$. In the following parts (a)-(c), we assume that α_i's are fixed effects, and $I = 3$.

(a) Under what conditions on w_i's, $\alpha_1 - \alpha_2$ is estimable? Is your condition necessary and sufficient? Explain clearly why.

(b) Assume that $w_1 = w_2 = w_3 \neq 0$. Consider the hypothesis testing for $H_0 : \alpha_1 = \alpha_2 = \alpha_3$ vs $H_1 : \alpha_i \neq \alpha_j$ for some $i \neq j \in \{1, 2, 3\}$. Construct a test statistic for testing H_0 vs H_1, give an α-level rejection region and provide the power function of the test.

(c) Consider a multiple testing problem for $H_{0j} : \alpha_j = \alpha_{j+1}$ vs $H_1 : \alpha_j \neq \alpha_{j+1}$ for $j = 1, 2$. Apply Scheffé's method to construct simultaneous test statistics and their rejection regions so that the family-wise type I error is controlled at the α-level.

In the following questions, we assume that α_i's are i.i.d. $N(0, \sigma_\alpha^2)$, α_i and ε_{ij} are independent, and $I = 2$.

(d) Write the mixed-effect model as a general mixed effects model of the form $\underline{y} = \mathbf{X}\underline{\beta} + \mathbf{Z}\underline{b} + \underline{\varepsilon}$, where $\underline{\beta}$ are fixed effects and $\underline{b}$ are random effects. Please specify $\mathbf{X}, \mathbf{Z}, \underline{\beta}$, and $\underline{b}$.

(e) Construct a test statistic and provide an α-level rejection region for testing $H_0 : \sigma_\alpha^2 = 0$ vs $H_0 : \sigma_\alpha^2 \neq 0$. Justify that your test procedure can maintain the type I error under the null hypothesis.

(f) Let us now assume that $w_i = i$ for $i = 1, 2$, and $\sigma_\alpha^2/\sigma^2 = 1$. Find the best linear unbiased predictor for $\alpha_1 - \alpha_2$.

(g) [Continued from part (f)] Construct an $1 - \alpha$ prediction interval for $\alpha_1 - \alpha_2$.

6. Consider the following linear mixed model:

$$Y_{ij} = \beta_0 + u_{0i} + (\beta_1 + u_{1i})t_{ij} + \epsilon_{ij},$$

where Y_{ij} is the response measurement for the ith individual's jth occasion taken at time t_{ij}. The coefficients β_0 and β_1 are fixed effects. Assume $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$,

$$\begin{pmatrix} u_{0i} \\ u_{1i} \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_0^2 & \sigma_{01} \\ \sigma_{01} & \sigma_1^2 \end{pmatrix} \right)$$

and $(u_{0i}\ u_{1i})'$ and ϵ_{ij} are independent.

(a) Derive the marginal variance $\text{Var}(Y_{ij})$ and covariance $\text{Cov}(Y_{ij}, Y_{ik})$, for $j \neq k$.

(b) What is the conditional variance $\text{Var}(Y_{ij}|u_{0i}, u_{1i})$?

(c) Suppose the observed responses for the first individual are $Y_{11} = 29$, $Y_{12} = 22$, $Y_{13} = 21$, $Y_{14} = 20$, and $Y_{15} = 28$, observed at days $t_{11} = 0$, $t_{12} = 2$, $t_{13} = 4$, $t_{14} = 6$, and $t_5 = 8$. Write the linear mixed model for this observation in the matrix notation:

$$\boldsymbol{Y_i} = \boldsymbol{X_i\beta} + \boldsymbol{Z_i u_i} + \boldsymbol{\epsilon_i}.$$