Preliminary Exam: Probability.

Time: 10:00am - 3:00pm, Friday, August 25, 2017.

Your goal should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: **If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!**

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.

2. Start each problem on a new page.

Prelim in Probability August 2017

Problem 1.

a. Let X, Y be identically distributed random variables with $P(X = k) = \frac{1}{3},\ k = -1, 0, 1.$

(i) Assume that X, Y are independent. Find the distribution of $X + Y$.

(ii). Find a joint distribution of X, Y so that $X + Y$ will have the same distribution as in part (i) but here X, Y are not independent. You can exhibit the joint distribution of X, Y in a 3x3 table.

b. For any random variable W we let $\varphi_W(t), t \in \mathcal{R}$ denote its characteristic function.

(i) Prove by using Fubini Theorem that if X, Y are independent then $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$, $t \in \mathcal{R}$

(ii) Is it true that $\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t),\ t \in \mathcal{R}$ implies that X, Y are independent? If your answer is positive then provide a proof; otherwise, provide a counterexample.

c. Use the multivariate inversion formula of characteristic function to prove that if

$\varphi_{X,Y}(s,t) = \varphi_X(s)\varphi_Y(t),\ s, t \in \mathcal{R}$ then X, Y are independent, where

$$\varphi_{X,Y}(s,t) \equiv E(\exp\{i(sX + tY)\}),\ s, t \in \mathcal{R} \text{ and } i = \sqrt{-1}\ .$$

Hint: Recall the inversion formula in the 2-dimensional case:

$$P\big((X,Y) \in A\big) = \lim_{T\to\infty} \frac{1}{(2\pi)^2} \int_{-T}^{T} \int_{-T}^{T} \psi(s,t)\varphi_{X,Y}(s,t)\,dsdt,$$

where $A = [a_1, b_1]x\ [a_2, b_2],\ a_i < b_i, i = 1, 2,$

$P(X = a_1) = P(X = b_1) = P(Y = a_2) = P(Y = b_2) = 0$, and

$$\psi(s,t) = \frac{(\exp(-isa_1) - \exp(-isb_1)) \cdot (\exp(-ita_2) - \exp(-itb_2))}{-st}$$

Problem 2

Let $\{W(t):\ 0 \le t \le 1\}$ denote a standard Brownian motion. In this problem $0 < s < u < t < 1$ are fixed. Denote $W(s,t) \equiv W(t) - W(s)$.

a. Let Y be a symmetric random variable, namely: $Y = -Y$, in distribution. Assume that $P(Y = 0) = 0$ (this assumption is only for the sake of simplicity)

(i) Prove that $\{\text{sign}(Y), |Y|\}$ are independent, where $\text{sign}(Y) \equiv \begin{cases} +1 & if\ \ Y > 0 \\ -1 & if\ \ Y < 0 \\ 0 & if\ \ Y = 0 \end{cases}$

(ii) Prove that: $\{\ W(s,u), |W(u,t)|, \text{sign}(W(u,t))\}$ are independent.

b. Let $\mathcal{G} \equiv\ \sigma\{W(s,u),\ |W(u,t)|\}$ and $\mathcal{F} \equiv \sigma\{|W(s,u)|,\ |W(u,t)|\}$.

(i) Find the conditional expectations $E_{\mathcal{G}}(W(s,u)W(u,t))$ and $E_{\mathcal{F}}(W(s,u)W(u,t))$.

(ii) Find $E_{\mathcal{F}}([W(s,t)]^2)$. Hint: $\sigma\{|X|\} = \sigma\{X^2\}$ for any random variable X.

c. Let $s \equiv u_0 < u_1 < \cdots < u_{d-1} < u_d \equiv t$. Find the conditional expectation: $E_{\mathcal{H}}(sign(W(s,t)))$, where $\mathcal{H} \equiv \sigma\{|W(u_0,u_1)|, |W(u_1,u_2)|, \ldots, |W(u_{d-1},u_d)|\}$.

Hints: Recall that $\{-W(u):\ 0 \le u \le 1\}$ is a standard Brownian motion as well. Also, $|(-W)(a,b)| = |W(a,b)|, 0 < a < b < 1$.

Remark. This problem leads to an application of backwards martingales to quadratic variation of Brownian motion.

Problem 3.

Let $(\mathcal{R}^2,\ \mathcal{B}(\mathcal{R}^2),\ P)$ be a probability space. Let $X(x,y) = x, Y(x,y) = y,\ (x,y) \in \mathcal{R}^2$ be the coordinate maps and let $\mathcal{F} = \sigma(X)$. We assume that (X,Y) has a joint density which is denoted by $f(x,y), (x,y) \in \mathcal{R}^2$ and let $f_X(x),\ x \in \mathcal{R}$ denote the density of X.

a. Let $A \in \mathcal{F}$ and let $\tilde{A} = \{X(a):\ a \in A\}$. Prove that $A = \tilde{A} \text{x}\, \mathcal{R}$.

b. Let $Z = z(X,Y)$ be a random variable where $z\colon \mathcal{R}^2 \to \mathcal{R}$ is measurable. We assume

$E(|Z|) < \infty$. Let H be a random variable defined by

$$H(x_0, y_0) = \frac{\int_{y=-\infty}^{\infty} z(x_0,y) f(x_0,y) dy}{f_X(x_0)},\ (x_0, y_0) \in \mathcal{R}^2.$$

Prove:

(i) $H \in \mathcal{F}$

(ii) $E(Z; A) = E(H; A),\ A \in \mathcal{F}$

(iii) What is the relationship between H and $E_{\mathcal{F}}(Z)$?

c. Let $\{P^{(x_0,y_0)}\colon (x_0, y_0) \in \mathcal{R}^2\}$ represent the **regular conditional probability** given $\mathcal{F}$.
Let $B \in \mathcal{B}(\mathcal{R}^2)$ and let $B_{x_0} = \{y \in \mathcal{R}:\ (x_0, y) \in B\}$. Express $P^{(x_0,y_0)}(B)$ by using $f(x,y)$, $f_X(x)$ and B_{x_0}.

Problem 4.

Let $\{ Z_k\},\ k = 1, 2, \ldots$ be identically distributed sequence of standard normal random variables.

a. Prove that for each $\epsilon > 0$: $\limsup_{k\to\infty} \frac{|Z_k|}{\sqrt{\ln(k)}} \leq \sqrt{2} + \epsilon$, a.s. [$\ln$ denotes the natural logarithm].

Hint: Recall the usual estimate for the tail of standard normal $P(|Z| > x) \leq e^{\frac{-x^2}{2}},\ x > 1$.

b. For the rest of the problem we assume that $\{ Z_k\}$ are also independent. Let $\{a_k\}_{k\geq 1}, \{b_k\}_{k\geq 1}$ be sequences of real numbers with $\sum_{k=1}^{\infty} a_k^2 + b_k^2 < \infty$.

(i) Prove that $\{\sum_{k=1}^{n} a_k Z_k, \mathcal{F}_n\}, n \geq 1$ is a martingale where $\mathcal{F}_n = \sigma\{Z_k, k = 1, \ldots, n\}$, and this martingale converges both a.s. and in L^2 to $\sum_{k=1}^{\infty} a_k Z_k$.

(ii) calculate $\mathrm{COV}(\sum_{k=1}^{\infty} a_k Z_k, \sum_{k=1}^{\infty} b_k Z_k)$.

(iii) Identify the distribution of $\sum_{k=1}^{\infty} a_k Z_k$ by using characteristic functions.

c. Let $\{s_k(t)\}_{k\geq 1}$ be a sequence of deterministic (non-random) functions defined on $[0,1]$. Assume that the process $B(t) \equiv \sum_{k=1}^{\infty} s_k(t) Z_k$, is well defined on $[0,1]$, in the sense that for each $t \in [0,1]$ the convergence of the random series is both a.s. and L^2.

(i). Express the $\mathrm{COV}(B(u), B(v))$, $0 \leq u, v \leq 1$, in terms of $\{s_k(t)\}_{k\geq 1}$. In particular, if our goal is that B will be a standard Brownian motion (ignoring continuity of sample paths for now) what should $\{s_k(t)\}_{k\geq 1}$ satisfy?

(ii) Let $s_k^* = \max_{0\leq t\leq 1}\{ | s_k(t)|\}$. Assume that $\sum_{k=1}^{\infty} s_k^* \sqrt{\ln(k)} < \infty$. Prove by using part a that

$$\sup_{0\leq t\leq 1} \{| \sum_{k=n}^{\infty} s_k(t) Z_k|\} \xrightarrow[n\to\infty]{} 0, \text{ a.s.}$$

Remark: This means that the convergence of $B(t) \equiv \sum_{k=1}^{\infty} s_k(t) Z_k$ is uniformly in t, a.s. If the functions $\{s_k(t)\}_{k\geq 1}$ are continuous then B has continuous sample paths.

Problem 5. Assume that $\{X_n\}_{n\geq 1}$ is uniformly integrable and that $X_n \to X, \text{a. s.}$ as $n \to \infty$. Let $\{\mathcal{F}_n\}_{n\geq 1}$ and $\mathcal{F}$ be σ-algebras.

a. Prove that $E(|X|) < \infty$.

b. Prove that $E_{\mathcal{F}_n}(X_n) \to E_{\mathcal{F}}(X)$ in L_1 as $n \to \infty$ (namely, $E(|E_{\mathcal{F}_n}(X_n) - E_{\mathcal{F}}(X)|) \to 0$ as $n \to \infty$) in the following 2 cases:

(i) if $\mathcal{F}_n \uparrow \mathcal{F}$

(ii) if $\mathcal{F}_n \downarrow \mathcal{F}$

Hint: $|a - b| \leq |a - c| + |c - b|$

c. Is it true that under the assumptions of this problem we also have: $E_{\mathcal{F}_n}(X_n) \to E_{\mathcal{F}}(X)$, a.s. as $\to \infty$? If your answer is positive then quote an appropriate theorem. If your answer is negative then modify our assumptions so that a.s. convergence will hold.

Problem 6. Let $\{X_{n,k}\}_{1 \le k \le n}$ and $\{Y_{n,k}\}_{1 \le k \le n}$, $n \ge 1$ be triangular arrays that are defined on the same probability space, For each fixed $n \ge 1$, the random variables $\{X_{n,k}, Y_{n,k}\}_{1 \le k \le n}$ are independent and $\{X_{n,k}\}_{1 \le k \le n}$ and $\{Y_{n,k}\}_{1 \le k \le n}$ are both identically distributed. Finally, let $S_n = \sum_{k=1}^{n} X_{n,k}$ and $T_n = \sum_{k=1}^{n} Y_{n,k}$, $n \ge 1$.

a. Assume $P\left(X_{n,1} = \frac{1}{\sqrt{n}}\right) = \frac{1}{2} = P\left(X_{n,1} = \frac{-1}{\sqrt{n}}\right)$. Prove that $S_n \to S$, in distribution and identify the distribution of S.

b. Let $P(Y_{n,1} = -1) = p_n,\ P(Y_{n,1} = 1) = q_n$ *and* $P(Y_{n,1} = 0) = 1 - p_n - q_n$. Also, we assume that: $np_n \to \lambda_1 > 0$ and $nq_n \to \lambda_2 > 0$, both limits as $n \to \infty$.

   (i) Write down $\varphi_{T_n}(t), t \in \mathcal{R}$ (the characteristic function of T_n)

   (ii) Prove that $\varphi_{T_n}(t) \to \varphi(t), t \in \mathcal{R}$ as $n \to \infty$. Hint: Use the lemma that deals with convergence of a sequence of products to an exponent.

   (iii) It follows from (ii) that $T_n \to T$, in distribution. Can you identify the distribution of T?

c. Does the sequence $\sum_{k=1}^{n} X_{n,k} + Y_{n,k}$ converges in distribution as $n \to \infty$? If your answer is positive then identify the limit distribution.