Preliminary Exam: Probability.

Time: 10:00am - 3:00pm, Friday, August 26, 2016.

Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: **If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!**

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.

2. Start each problem on a new page.

3. Write only on one side of a page.

Problem 1. Let X, Y be random variables defined on the same probability space. We assume $E(X^2) + E(Y^2) < \infty$. Let $X_{a,b} \equiv a + bX$, where $(a, b) \in R^2$.

a. Express $(a^*, b^*) \in R^2$ in terms of $E(X), E(Y), Var(X), Var(Y), Cov(X, Y)$, where (a^*, b^*) satisfy

$$\begin{cases} E\left(Y - X_{a^*,b^*}\right) = 0 \\ E\left(\left(Y - X_{a^*,b^*}\right)X\right) = 0. \end{cases}$$

b. (i) Calculate $E[(Y - X_{a^*,b^*})(X_{a^*,b^*} - X_{a,b})]$, where $(a, b) \in R^2$ and (a^*, b^*) is defined in part a.

(ii) Prove by using (i): $\min\limits_{(a,b)\in R^2} E\left[\left(Y - X_{a,b}\right)^2\right] = E\left[\left(Y - X_{a^*,b^*}\right)^2\right]$.

c. Let $\mathcal{X}$ be a family of random variables defined by $\mathcal{X} \equiv \{\text{W: } E(W^2) < \infty \text{ and } W \in \sigma\{X\}\}$, where $\sigma\{X\}$ is the σ –algebra generated by X. Calculate $E[(Y - E_{\sigma\{X\}}(Y))(E_{\sigma\{X\}}(Y) - W)]$, where $W \in \mathcal{X}$. Use this calculation to find $W^* \in \mathcal{X}$ that solves :

$\min\limits_{W\in\mathcal{X}} E[(Y - W)^2] = E[(Y - W^*)^2]$.

Problem 2. Let $\{X, X_m\colon\ m = 1,2, \dots\}$ be a sequence of independent and identically distributed random variables. Let $S_n = \sum_{m=1}^{n} X_m,\ n \geq 1$.

a. Let $b_n > 0,\ a_n = n \cdot E(X\colon |X| \leq b_n),\ n \geq 1$. Prove for each $\varepsilon > 0, n \geq 1$

$$P\left(\left|\frac{S_n - a_n}{b_n}\right| > \varepsilon\right) \leq nP(|X| \geq b_n) + \frac{n \cdot E\left(X^2\colon |X| \leq b_n\right)}{\varepsilon^2 b_n^2} \quad .$$

b. Assume that the distribution of X is specified by $P\left(X = 2^k\right) = 2^{-k},\ k \geq 1$.

Use part a to show that

$$\frac{S_n}{n \cdot \log_2(n)} \to 1 \text{ in probability as } n \to \infty.$$

Hint: In order to streamline the calculations you may use , without proof, the following 2 inequalities that hold for each $x > 0$:

(i) $\sum_{k=1}^{\infty} 2^{-k} 1_{\{2^k > x\}} \leq \frac{2}{x}$, (ii) $\sum_{k=1}^{\infty} 2^{k} 1_{\{2^k < x\}} \leq 2x$.

Problem 3. Let $\{X_k: \ k = 1,2,\dots\}$ be a sequence of independent random variables. We assume that $E(X_k) = 0$, and $E[(X_k^-)^2] < \infty, k = 1,2,\dots$where $X_k^- = \max\{-X_k, 0\}, X_k^+ = \max\{X_k, 0\}$.

a. Prove that $E\left(\exp\left\{X_k - \frac{(X_k^+)^2}{2}\right\}\right) \le 1 + \frac{E[(X_k^-)^2]}{2}, \ k = 1,2,\dots$

Hint: You may want to use the following inequalities without proving them

(i) $\qquad \exp\{y\} \le 1 + y + \frac{y^2}{2}, \ y < 0$

(ii) $\qquad \exp\left\{y - \frac{y^2}{2}\right\} \le 1 + y, \ y > 0$

b. Prove that $E\left(\exp\left\{X_k - \frac{(X_k^+)^2 + E[(X_k^-)^2]}{2}\right\}\right) \le 1\ , \ k = 1,2,\dots$

c. Prove that $\{\exp\{S_n - \frac{W_n^2}{2}\}, \ \mathcal{F}_n\}, n = 1,2,\dots$ is a super-martingale, where $S_n = \sum_{k=1}^n X_k, \ n \ge 1, \ W_n^2 \equiv \sum_{k=1}^n (X_k^+)^2 + \sum_{k=1}^n E[(X_k^-)^2$ and $\mathcal{F}_n = \sigma\{X_k: \ k = 1,\dots,n\}$.

Problem 4. Let $\{X_m: \ m = 1,2,\dots\}$ be a sequence of independent and symmetric random variables (i.e. $X_m = -X_m, \ m \ge 1,$ in distribution). Let $S_n = \sum_{m=1}^n X_m, \ n \ge 1.$

a. (i) Prove that any random variable X is symmetric if and only if its characteristic function is real valued.

(ii) Prove that $S_n, n \ge 1$ is symmetric.

For the rest of the problem we assume that $S_n \to S$ in probability as $n \to \infty$. Also, all the statements in what follows are for each $\varepsilon > 0$.

b. Use Levy inequality (Recall: $P\left(\max_{1\le k\le n} |S_k| > t\right) \le 2P(|S_n| > t), t > 0, n \ge 1)$ in order to prove: $P(\sup_{n>M}\{|S_n - S_M|\} > \varepsilon) \to 0$ as $M \to \infty$.

Hint: Prove first that $\sup_{n>M} P(|S_n - S_M| > \varepsilon) \to 0$ as $M \to \infty$.

c. Show that in fact $S_n \to S$ a.s. as $n \to \infty$.

Hint: Show first that part b implies: $P(\sup_{n,m>M}\{|S_n - S_m|\} > \varepsilon) \to 0$ as $M \to \infty$. Then consider the sequence $a_M \equiv \sup_{n,m>M}\{|S_n - S_m|\}, \ M = 1,2,\dots$ Is it monotone a.s. ?

Problem 5. Let $\{X_{n,k}:\ 1 \le k \le n, n = 1, \dots\}$ and be a triangle arrays of row-wise independent random variables with $E(X_{n,k}) = 0$. We assume:

(i) For each $\varepsilon > 0$, $\sum_{k=1}^{n} E\left(X_{n,k}^2:\ |X_{n,k}| > \varepsilon\right) \to 0$ as $n \to \infty$

(ii) There exist non random $0 = t_0 < t_1 < \cdots < t_d = 1$ so that for each $1 \le m \le d$

$\sum_{k=1}^{[nt_m]} E\left(X_{n,k}^2\ \right) \to t_m$,as $n \to \infty$, where $[t]$ is the largest integer that is less or equal to t.

a. Prove that for each $1 \le m \le d$:

$$\sum_{[nt_{m-1}]+1}^{[nt_m]} X_{n,k} \to N(0, t_m - t_{m-1}) \quad ,\text{as } n \to \infty$$

b. Prove the convergence in distribution, as $n \to \infty$, of the following d- dimensional random vectors: $(\sum_{[nt_{m-1}]+1}^{[nt_m]} X_{n,k}:\ m = 1, \dots, d)$. What is the characteristic function of the limit distribution?

c. Prove that $(\sum_{1}^{[nt_m]} X_{n,k}:\ m = 1, \dots, d) \to (W(t_m):\ m = 1, \dots, d)$ in distribution as $n \to \infty$, where $\{W(t):\ t \ge 0\}$ is a standard Brownian motion.

Problem 6. Let μ, γ be 2 probability measures on the space ((0, 1], $\mathcal{B}$) where $\mathcal{B}$ denote the Borel $\sigma -$ algebra on $(0, 1]$. Let $\rho = \frac{\mu+\gamma}{2}$ and assume that $\rho(I_{n,k}) > 0$, where $I_{n,k} \equiv \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]$. **Here and throughout** $k = 1, \dots, 2^n, n = 0, 1, \dots$. Let $X_n(t) \equiv \frac{\mu(I_{n,k})}{\rho(I_{n,k})}$ if $t \in I_{n,k}$. **Prove the following**:

a. (i) $0 \le X_n \le 2,\ n \ge 1$

(ii) $\{X_n, \mathcal{F}_n\}, n \ge 1$ is a martingale on ((0, 1], $\mathcal{B}, \rho$) , where $\mathcal{F}_n \equiv \sigma\{I_{n,k}: k = 1, \dots, 2^n\}$.

b. (i) $X = \lim_{n\to\infty} X_n$ exists a.s. with respect to ρ.

(ii) $\mu(A) = \int_A X d\rho,\ A \in \mathcal{B}$.

Hint: Prove (ii) first for events of the type $A = I_{n,k}$, then for $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$ and finally extend the result to $A \in \mathcal{B}$ by using the $\pi - \lambda$ theorem.

c. (i) $\int_A (2 - X) d\mu = \int_A X d\gamma,\ A \in \mathcal{B}$.

(ii) $\gamma(\{X = 2\}) = 0$.