Large Deviations for Translation Invariant Functionals of Brownian Occupation Times

S.R.S. Varadhan
New York University

Michigan State University
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Large Deviations.
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\[ X, \{P_\epsilon\}. \quad P_\epsilon \to \delta_x \text{ as } \epsilon \to 0. \]
Large Deviations.

\( X, \{P_\varepsilon\} \). \( P_\varepsilon \to \delta_x \) as \( \varepsilon \to 0 \).

\[
P_\varepsilon(A) \simeq \exp\left[-\frac{1}{\varepsilon} \inf_{x \in A} I(x) + o\left(\frac{1}{\varepsilon}\right)\right]
\]
Large Deviations.

$X, \{P_\epsilon\}$. $P_\epsilon \to \delta_x$ as $\epsilon \to 0$.

$$P_\epsilon(A) \simeq \exp\left[-\frac{1}{\epsilon} \inf_{x \in A} I(x) + o\left(\frac{1}{\epsilon}\right)\right]$$

Lower bound for open sets and upper bound for closed sets.
Conclusions.
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\[ \epsilon \log \int \exp \left[ \frac{F(x)}{\epsilon} \right] dP_\epsilon \to \sup_x [F(x) - I(x)] \]
Conclusions.

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\[ dQ_\epsilon = [Z_\epsilon]^{-1} \exp\left[ \frac{F(x)}{\epsilon} \right] dP_\epsilon \]
Conclusions.

\[ \epsilon \log \int \exp\left[ \frac{F(x)}{\epsilon} \right] dP_\epsilon \rightarrow \sup x \left[ F(x) - I(x) \right] \]

Where

\[ dQ_\epsilon = [Z_\epsilon]^{-1} \exp\left[ \frac{F(x)}{\epsilon} \right] dP_\epsilon \]

\[ Z_\epsilon = \int \exp\left[ \frac{F(x)}{\epsilon} \right] dP_\epsilon \]
$Q_\epsilon \to \delta_y$
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\( y \) is the unique point such that
\[
F(y) - I(y) = \sup_x [F(x) - I(x)].
\]
$Q_\epsilon \rightarrow \delta_y$

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For the proof to work out one needs the upper bound to hold for all closed sets.
\( Q_\epsilon \rightarrow \delta_y \)

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- Often one gets local estimates. Pushed to Compact sets.
$Q_{\epsilon} \to \delta_y$

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Often one gets local estimates. Pushed to Compact sets.

$$e^{\frac{a}{\epsilon}} + e^{\frac{b}{\epsilon}} = e^{\frac{\max\{a,b\}}{\epsilon}} + o\left(\frac{1}{\epsilon}\right)$$
Markov process. ergodic theorem. Brownian motion on the circle.
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Normalized Lebesgue measure is invariant
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Normalized Lebesgue measure is invariant

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- Normalized Lebesgue measure is invariant

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\( Q_T \) is the distribution of \( L_T \)
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\( Q_T \) is the distribution of \( L_T \)

\[ \int \exp[T \int V(x) \mu(dx)] Q_T(d\mu) = E[\exp \int_0^T V(x(s)) \, ds] \]
$Q_T$ has a LDP. \( d\mu = f(x)dx \)
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$I(\mu) = \frac{1}{8} \int_{S^1} \frac{[f'(x)]^2}{f(x)} dx$
\( Q_T \) has a LDP. \( d\mu = f(x)dx \)

\[ I(\mu) = \frac{1}{8} \int_{S^1} \left[ \frac{f'(x)}{f(x)} \right]^2 dx \]

\( \mathcal{M}(S^1) \) is compact.
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If we replace $S^1$ by $R$, there is a problem.
- $Q_T$ has a LDP. $d\mu = f(x)dx$
- $I(\mu) = \frac{1}{8} \int_{S^1} \frac{[f'(x)]^2}{f(x)} dx$
- $\mathcal{M}(S^1)$ is compact.
- If we replace $S^1$ by $R$, there is a problem.
- There is no invariant measure. dissipative.
Can remedy it by compactifying $\mathbb{R}$ by adding $\infty$. 
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Then $M$ becomes $M_{\leq 1}$. 
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$I(0) = 0$. 
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$\int V(x) \, d\mu$ with $V(x) \to 0$ as $|x| \to \infty$ are OK.
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One can compactify space add point at $\infty$.

The missing mass is at $\infty$. 
The rate function is still the same.
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\[ I(f + c\delta_{\infty}) = I(f) \]
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\[ Z_T = E\left[ \exp \left( \frac{1}{T} \int_0^T \int_0^T V(x(s) - x(t)) \, ds \, dt \right) \right] \]
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\[ = \sup_f \left[ \int \int V(x - y) f(x) f(y) dx dy - I(f) \right] \]
Brownian Motion in $\mathbb{R}^3$. 
Brownian Motion in $R^3$.

$$\psi_T(\omega) = \frac{1}{T} \int_0^T \int_0^T \frac{1}{\left| x(t) - x(s) \right|} dsdt$$
Brownian Motion in $R^3$.

$$\psi_T(\omega) = \frac{1}{T} \int_0^T \int_0^T \frac{1}{|x(t) - x(s)|} ds dt$$

$$\psi_T(\omega) = T \int \int \frac{1}{|x - y|} L_T(dx)L_T(dy)$$
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\[
Z_T = E[\exp[\psi_T(\omega)]]
\]

\[
dQ_T = \frac{1}{Z_T} \exp[\psi_T(\omega)] dP
\]
The problem is translation invariant.
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- Natural space is $\mathcal{X} = \mathcal{M}(\mathbb{R}^3)/\mathbb{R}^3$
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$P[L_T \sim f \, dx] = \exp[-TI(f)]$
The problem is translation invariant.

Natural space is \( \mathcal{X} = \mathcal{M}(R^3)/R^3 \)

Not compact.

There is local LDP

\[
P[L_T \simeq f \, dx] = \exp[-TI(f)]
\]

\[
I(f) = \frac{1}{8} \int \frac{\|\nabla f\|^2}{f^2} dx
\]
The variational problem

\[
\sup_f \left[ \int \frac{1}{|x-y|} f(x)f(y) \, dx \, dy - I(f) \right]
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Has a unique maximizer. (modulo translation)
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One expects on $\tilde{\mathcal{X}} = \mathcal{X}/R^3$, $Q_T \rightarrow \delta_{\tilde{f}}$
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Compactify \( \tilde{\mathcal{X}} \)
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Compactify \( \tilde{\mathcal{X}} \)

Identify the compactification.
The variational problem

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Prove the upper and lower bounds at the new points.
The variational problem
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Compactify \( \tilde{\mathcal{X}} \)

Identify the compactification.

Prove the upper and lower bounds at the new points.

Show the supremum now is still attained at the same \( f \).
Joint work with Chiranjib Mukherjee.
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$F(L_T)$. 
Joint work with Chiranjib Mukherjee.

\[ F(L_T). \]

\[ F(\mu) = F(\mu \ast \delta_a) \]
Joint work with Chiranjib Mukherjee.

\[ F(L_T) \]

\[ F(\mu) = F(\mu * \delta_a) \]

Examples
Joint work with Chiranjib Mukherjee.

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Examples

$F(\mu) = \int_{(\mathbb{R}^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k)$
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Examples

$F(\mu) = \int_{R^3} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k)$

$f(x_1 + x, \ldots, x_k + x) = f(x_1, x_2, \ldots, x_k)$
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$f \to 0$ if $\sup_{i,j} |x_i - x_j| \to \infty$
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f \to 0 \text{ if } \sup_{i,j} |x_i - x_j| \to \infty

\frac{1}{T} \log E[\exp[TF]]
How to compactify?
- How to compactify?
- One point compactification is not suitable.
How to compactify?

- One point compactification is not suitable.
- Is not translation invariant.
How to compactify?

One point compactification is not suitable.

Is not translation invariant.

The unboundedness of $\frac{1}{|x|}$ is not a problem.
Take a function $f(x_1, \ldots, x_k)$ that is translation invariant and continuous.
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Tends to 0 if $\sup_{i,j} |x_i - x_j| \to \infty$
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$\Lambda(f, \mu) = \int_{(R^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k)$
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- Countable collection $\{f_j\}$ is enough.
Take a function \( f(x_1, \ldots, x_k) \) that is translation invariant and continuous.

Tends to 0 if \( \sup_{i,j} |x_i - x_j| \to \infty \)

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\( \Lambda(f, \mu) = \int_{(\mathbb{R}^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) \)

Countable collection \( \{f_j\} \) is enough.

\( \mathcal{F}_{k-1} \) can be obtained from \( \mathcal{F}_k \)
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Tends to 0 if $\sup_{i,j} |x_i - x_j| \to \infty$

$\mathcal{F} = \bigcup_k \mathcal{F}_k$

$\Lambda(f, \mu) = \int_{(\mathbb{R}^3)^k} f(x_1, x_2, \ldots, x_k)\mu(dx_1) \cdots \mu(dx_k)$

Countable collection $\{f_j\}$ is enough.

$\mathcal{F}_{k-1}$ can be obtained from $\mathcal{F}_k$

$f_k(x_1, \ldots, x_k) = f_{k-1}(x_1, \ldots, x_{k-1})\phi(x_1 - x_k)$
Take a function \( f(x_1, \ldots, x_k) \) that is translation invariant and continuous.

Tends to 0 if \( \sup_{i,j} |x_i - x_j| \to \infty \)

\[ \mathcal{F} = \bigcup_k \mathcal{F}_k \]

\[ \Lambda(f, \mu) = \int_{(R^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) \]

Countable collection \( \{f_j\} \) is enough.

\( \mathcal{F}_{k-1} \) can be obtained from \( \mathcal{F}_k \)

\[ f_k(x_1, \ldots, x_k) = f_{k-1}(x_1, \ldots, x_{k-1}) \phi(x_1 - x_k) \]

\[ \int f_k \Pi \mu(dx_i) \to \mu(R^3) \int f_{k-1} \Pi \mu(dx_i) \]
Choose a subsequence so that
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\[ \lim_{n \to \infty} \Lambda(f, \mu_n) = \lambda(f) \text{ exists for } f \in \mathcal{F}. \]
What is \( \lambda(f) \).
Choose a subsequence so that
\[ \lim_{n \to \infty} \Lambda(f, \mu_n) = \lambda(f) \] exists for \( f \in \mathcal{F} \).
What is \( \lambda(f) \).
Trying to complete with the metric
\[
D(\mu_1, \mu_2) = \sum c_j |\Lambda(f_j, \mu_1) - \Lambda(f_j, \mu_2)|
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\[ c_j = \frac{1}{2^j} \frac{1}{1 + \|f_j\|_\infty} \]
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where
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\]
and
\[
\xi = \{\tilde{\mu}\}, \sum_{\tilde{\mu} \in \xi} \mu(R^3) = p \leq 1
\]
\[ \Lambda(\xi, f) = \sum_{\tilde{\mu} \in \xi} \int_{(\mathbb{R}^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) \]
\[ \Lambda(\xi, f) = \sum_{\tilde{\mu} \in \xi} \int_{(R^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) \]

How can a sequence \( \mu_n \) fail to be compact?
\[ \Lambda(\xi, f) = \sum_{\tilde{\mu} \in \xi} \int_{(\mathbb{R}^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) \]

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How can a sequence \( \mu_n \) fail to be compact?

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\( \mu_n = \frac{1}{2} [\mu \ast \delta_{a_n} + \mu \ast \delta_{-a_n}] \)
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\[ \mu_n = \mathcal{N}(0, nI) \]
\[ \Lambda(\xi, f) = \sum_{\tilde{\mu} \in \xi} \int_{(R^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) \]

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- \( \mu_n = \frac{1}{2} [\mu \ast \delta_{a_n} + \mu \ast \delta_{-a_n}] \)
- \( \mu_n = N(0, nI) \)

The orbit converges.
\[
\Lambda(\xi, f) = \sum_{\tilde{\mu} \in \xi} \int_{(R^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k)
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\( \mu_n = N(0, nI) \)

The orbit converges.

The limit is in two pieces. \( \mu_1, \mu_2 \) of mass \( \frac{1}{2} \) each.
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\Lambda(\xi, f) = \sum_{\tilde{\mu} \in \xi} \int_{(R^3)^k} f(x_1, x_2, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k)
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Becomes dust.
Compactification $\tilde{\mathcal{X}}$. 
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- Collection of orbits $\xi = \{\tilde{\mu}_\alpha\}$
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$\sum_\alpha \mu(R^3) = \sum p_\alpha = p \leq 1$

Empty, finite or countable.
Compactification $\tilde{\mathcal{X}}$.

Collection of orbits $\tilde{\xi} = \{\tilde{\mu}_\alpha\}$

$$\sum_\alpha \mu(R^3) = \sum p_\alpha = p \leq 1$$

Empty, finite or countable.

Have a metric.
\[ \Lambda(f, \xi) = \sum_{\tilde{\mu} \in \xi} \int f(x_1, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) \]
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\[ D(\xi_1, \xi_2) = \sum c_r |\Lambda(f_r, \xi_1) - \Lambda(f_r, \xi_2)| \]
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Does \[ \Lambda(f, \xi_1) = \Lambda(f, \xi_2), \forall f \]

imply \( \xi_1 = \xi_2 \)?
\[ g_N(x_1, x_2, \ldots, x_{2k}) = f(x_1, \ldots, x_k) f(x_{k+1}, \ldots, x_{2k}) \phi_N(x_k - x_{2k}) \]
\[ g_N(x_1, x_2, \ldots, x_{2k}) = \\
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\[ \Lambda(g_N, \xi) \rightarrow \\
\sum_{\tilde{\mu} \in \xi} \left[ \int f(x_1, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) \right]^2 \]
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Does it mean we know \( \xi \)?

Let \( \xi_1 \) and \( \xi_2 \) be two collections such that for every \( f \), \( \{ \int f(x_1, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) \} \) are the same as \( \tilde{\mu} \) varies over \( \xi_1 \) or \( \xi_2 \).
Is $\xi_1 = \xi_2$?
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Given $\tilde{\mu} \in \xi_1$ consider for $\tilde{\nu} \in \xi_2$,

$$C_{\tilde{\nu}} = \{ f \in F_k : \Lambda(f, \tilde{\mu}) = \Lambda(f, \tilde{\nu}) \}$$
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All the choices for different $k$ have same mass.
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Finite number.
One of them has $k >> 1$
- One of them has $k >> 1$
- $\forall f \in \mathcal{F}_k$ and $\forall k \geq 2$
One of them has $k \gg 1$

$\forall f \in \mathcal{F}_k$ and $\forall k \geq 2$

$$\int f(x_1, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k) =$$
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Does it imply $\mu = \nu \ast \delta_a$ for some $a$?
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$\phi = \hat{\mu}(t), \psi = \hat{\nu}(t)$
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\[ \phi = \hat{\mu}(t), \psi = \hat{\nu}(t) \]

\[ \prod_{i=1}^{k} \phi(t_i) = \prod_{i=1}^{k} \psi(t_i) \text{ if } \sum_i t_i = 0. \]
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$\forall f \in \mathcal{F}_k$ and $\forall k \geq 2$

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$\phi = \hat{\mu}(t), \psi = \hat{\nu}(t)$

$\pi_{i=1}^k \phi(t_i) = \pi_{i=1}^k \psi(t_i)$ if $\sum_i t_i = 0$.

$\phi(t)\phi(-t) = \psi(t)\psi(-t)$
\[ |\phi(t)| = |\psi(t)| \]
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\[ \phi(t) = \psi(t)\chi(t) \] on \( G = \{t : |\phi(t)| \neq 0\} \)
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\[ \chi(nt) = [\chi(t)]^n, \chi(t) = e^{i<a,t>} \]
\( \tilde{\mathcal{M}} \) is dense in \( \tilde{\mathcal{X}} \)
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\( q_\mu(r) = \sup_x \mu [B(x, r)] \)
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Depends only on the orbit.
$q = 1. \mu_n$ is tight after translation.
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- $q = 0$ disintegrates to dust. tends to $\xi = 0$.
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- Repeat and exhaust.
Local upper bounds about the new points in $\tilde{X}$. 
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Lower bound is easy.
- Local upper bounds about the new points in $\tilde{X}$.
- Lower bound is easy.
- $\tilde{\mu}_n \rightarrow \xi$ with $I(\mu_n) \rightarrow I(\xi) = \sum_{\tilde{\mu} \in \xi} I(\mu_j)$
Local upper bounds about the new points in $\tilde{X}$.

Lower bound is easy.

$\tilde{\mu}_n \rightarrow \xi$ with $I(\mu_n) \rightarrow I(\xi) = \sum_{\tilde{\mu} \in \xi} I(\mu_j)$

$I(\mu) = \sup_{u>0} \left[ - \int \frac{1}{2} \frac{\Delta u}{u} d\mu \right]$
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$\exp \left[ - \int_0^t \frac{1}{2} \Delta u \frac{1}{u} (x(s)) ds \right] \leq \frac{\sup_x u(x)}{\inf_x u(x)}$
$\nu$ compact support, smooth. $u = \nu + c$
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g(k, \ell, c, a_1, \ldots, a_k, x) = c + \sum_{i=1}^{k} u_i(x + a_i) \phi\left(\frac{x + a_i}{\ell}\right)$
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$F(u_1, \ldots, u_k, c, \ell, t, \omega)$
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$F(u_1, \ldots, u_k, c, \ell, t, \omega)$

$$\sup_{a_1, \ldots, a_k} \inf_{i \neq j} \frac{1}{t} \int_{0}^{t} -\frac{1}{2} \Delta g(k, \ell, c, a_1, \ldots, a_k, x(s)) \frac{1}{g(k, \ell, c, a_1, \ldots, a_k, x(s))} ds$$
\[
\sup_{a_1, \ldots, a_k} \inf_{i \neq j} \left| a_i - a_j \right| \geq 4\ell \int_d \frac{-\frac{1}{2} \Delta g(k, \ell, c, a_1, \ldots, a_k, x)}{g(k, \ell, c, a_1, \ldots, a_k, x)} L_t(dx)
\]
\[ \sup_{a_1, \ldots, a_k} \inf_{i \neq j} \left| a_i - a_j \right| \geq 4 \ell \int d \frac{-\frac{1}{2} \Delta g(k, \ell, c, a_1, \ldots, a_k, x)}{g(k, \ell, c, a_1, \ldots, a_k, x)} \tilde{F}(u_1, \ldots, u_k, c, \ell, \tilde{L}_t) \]
\[
\sup_{a_1, \ldots, a_k} \inf_{i \neq j} \frac{1}{|a_i - a_j|} \int d \frac{-\frac{1}{2} \sum_{k, \ell, c, a_1, \ldots, a_k, x} g(k, \ell, c, a_1, \ldots, a_k, x)}{L_t(dx)}
\]

\[
\widetilde{F}(u_1, \ldots, u_k, c, \ell, \widetilde{L}_t)
\]

\[
E \left[ \exp \left[ \int_0^t \frac{-\frac{1}{2} \Delta g(x(s))}{g(x(s))} ds \right] \right] \leq \frac{C}{c}
\]
Small variations in $a_i$ change little.
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$|a_i| \leq t^2$?
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- sup over polynomially many sets of $\{a_i\}$. 
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- $|a_i| \leq t^2$?
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- $u_{i, \ell} = u_i(x) \phi(\frac{x}{\ell})$
\[
\liminf_{\mu \to \xi} \tilde{F}(u_1, \ldots, u_k, c, \ell, \tilde{\mu}) \geq \]

\[
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\sum_{i=1}^{k} \int \frac{-\left(\frac{1}{2} \Delta u_{i,\ell}(x)\right)}{c + u_{i,\ell}(x)} \alpha_i(d\gamma)
\]
\[
\liminf_{\mu \to \xi} \tilde{F}(u_1, \ldots, u_k, c, \ell, \tilde{\mu}) \geq \\
\sum_{i=1}^{k} \int \frac{-\left(\frac{1}{2} \Delta u_{i,\ell}(x)\right)}{c + u_{i,\ell}(x)} \alpha_i(dx) \\
\Lambda(\xi, \ell, c, u_1, \ldots, u_k)
\]
$\lim inf_{\mu \to \xi} \tilde{F}(u_1, \ldots, u_k, c, \ell, \tilde{\mu}) \geq$

$$\sum_{i=1}^{k} \int \frac{-(\frac{1}{2} \Delta u_{i,\ell})(x)}{c + u_{i,\ell}(x)} \alpha_i(dx)$$

$\Lambda(\xi, \ell, c, u_1, \ldots, u_k)$

$$\sup_{c,k,\ell,u_1,\ldots,u_k} \Lambda(\xi, \ell, c, u_1, \ldots, u_k) = \tilde{I}(\xi)$$
\[
\lim \inf_{\mu \to \xi} \tilde{F}(u_1, \ldots, u_k, c, \ell, \tilde{\mu}) \geq \\
\sum_{i=1}^{k} \int \frac{-\left( \frac{1}{2} \Delta u_i, \ell \right)(x)}{c + u_i, \ell(x)} \alpha_i(dx) \\
\Lambda(\xi, \ell, c, u_1, \ldots, u_k) \\
\sup_{c, k, \ell, u_1, \ldots, u_k} \Lambda(\xi, \ell, c, u_1, \ldots, u_k) = \tilde{I}(\xi) \\
\tilde{I}(\xi) = \sum_{\tilde{\mu} \in \xi} I(\mu)
\]
\[ F(\mu) = \int \frac{1}{|x_1 - x_2|} \mu(dx_1) \mu(dx_2) \]
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- Singularity is not a problem.
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*Singularity is not a problem.*

*Variational problem is*

\[
\sup_\xi \left[ \Lambda\left(\frac{1}{|x - y|}, \xi\right) - I(\xi) \right]
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- Sup is attained at \( \xi = \{\tilde{\mu}_0\} \), a single orbit of unit mass.
$$F(\mu) = \int \frac{1}{|x_1 - x_2|} \mu(dx_1) \mu(dx_2)$$

- Singularity is not a problem.
- Variational problem is

$$\sup_{\xi} \left[ \Lambda\left( \frac{1}{|x - y|}, \xi \right) - I(\xi) \right]$$

- Sup is attained at $\xi = \{\tilde{\mu}_0\}$, a single orbit of unit mass.
- Unique up to translation. On $\tilde{X}$ there is a unique maximum.
The mass under $Q_T$ concentrates in a neighborhood of the orbit.
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$Q_T \Rightarrow \delta_{\tilde{\mu}_0}$
Thank You