Large Deviations for Translation Invariant Functionals of Brownian Occupation Times

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- Lower bound for open sets and upper bound for closed sets.

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- $\int \exp[T \int V(x)\mu(dx)]Q_T(d\mu) = E[\exp \int_0^T V(x(s))ds]$

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- There is no invariant measure. dissipative.

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- Show the supremum now is still attained at the same f.

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- $\xi = \{\widetilde{\mu}\}, \sum_{\widetilde{\mu}\in\xi} \mu(R^3) = p \leq 1$

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- Does
$$\Lambda(f,\xi_1) = \Lambda(f,\xi_2), \forall f$$
imply $\xi_1 = \xi_2$?

- $g_N(x_1, x_2, \ldots, x_{2k}) =$
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- Let ξ_1 and ξ_2 be two collections such that for every f, $\{\int f(x_1, \ldots, x_k) \mu(dx_1) \cdots \mu(dx_k)\}$ are the same as $\tilde{\mu}$ varies over ξ_1 or ξ_2.

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- Repeat and exhaust.

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- $\exp\left[- \int_0^t \frac{\frac{1}{2}\Delta u}{u}(x(s))ds \right] \leq \frac{\sup_x u(x)}{\inf_x u(x)}$

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- $$\sup_{\substack{a_1,\ldots a_k \\ \inf_{i \neq j} |a_i - a_j| \geq 4\ell}} \frac{1}{t} \int_0^t \frac{-\frac{1}{2}\Delta g(k, \ell, c, a_1, \ldots, a_k, x(s))}{g(k, \ell, c, a_1, \ldots, a_k, x(s))} ds$$

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Thank You