Scaling Limits

S.R.S. Varadhan
Courant Institute, NYU

Michigan State University
April 1, 2015
We model phenomena at a very small (micro) scale.
We model phenomena at a very small (micro) scale.
Because that is where we understand the phenomenon and can make a reasonable model.
We model phenomena at a very small (micro) scale. 
Because that is where we understand the phenomenon and can make a reasonable model. 
We are interested in answers to question that are posed on a large (macro) scale.
We model phenomena at a very small (micro) scale.

Because that is where we understand the phenomenon and can make a reasonable model.

We are interested in answers to question that are posed on a large (macro) scale.

How do we make the transition?
We model phenomena at a very small (micro) scale.

Because that is where we understand the phenomenon and can make a reasonable model.

We are interested in answers to question that are posed on a large (macro) scale.

How do we make the transition?

Depends on the context.
We model phenomena at a very small (micro) scale.

Because that is where we understand the phenomenon and can make a reasonable model.

We are interested in answers to questions that are posed on a large (macro) scale.

How do we make the transition?

Depends on the context.

Some times it is straightforward.
We model phenomena at a very small (micro) scale. Because that is where we understand the phenomenon and can make a reasonable model.

We are interested in answers to questions that are posed on a large (macro) scale.

How do we make the transition?

Depends on the context.

Some times it is straightforward.

Some times it is very complicated.
We model phenomena at a very small (micro) scale. Because that is where we understand the phenomenon and can make a reasonable model.

We are interested in answers to question that are posed on a large (macro) scale.

How do we make the transition?

Depends on the context.

Some times it is straight forward.

Some times it is very complicated

We will look at some examples.
Consider the lattice $\mathbb{Z}_h^d$ with spacing $h$. 
Consider the lattice $Z_h^d$ with spacing $h$.

$G \subset R^d$ with boundary $\partial G$
Consider the lattice $Z_h^d$ with spacing $h$.

$G \subset R^d$ with boundary $\partial G$

$$\sum_{y : y \approx x} [f_h(y) - f_h(x)] = 0 \quad x \in G$$
Consider the lattice $\mathbb{Z}_h^d$ with spacing $h$.

$G \subset \mathbb{R}^d$ with boundary $\partial G$

$$\sum_{y : y \preceq x} [f_h(y) - f_h(x)] = 0 \quad x \in G$$

$f_h(y) = g(y)$ is given for $y \not\in G, y \preceq G$. 
Consider the lattice $\mathbb{Z}^d_h$ with spacing $h$.

$G \subset \mathbb{R}^d$ with boundary $\partial G$

$$\sum_{y : y \approx x} [f_h(y) - f_h(x)] = 0 \quad x \in G$$

$f_h(y) = g(y)$ is given for $y \notin G, y \approx G$.

As $h \to 0$ we get $f_h \to f$

$$\Delta f = 0, \quad x \in G; \text{ and } f|_{\partial G} = g$$
Consider the lattice $\mathbb{Z}_h^d$ with spacing $h$.  
$G \subset \mathbb{R}^d$ with boundary $\partial G$

$$\sum_{y : y \simeq x} [f_h(y) - f_h(x)] = 0 \quad x \in G$$

$f_h(y) = g(y)$ is given for $y \notin G, y \simeq G$.

As $h \to 0$ we get $f_h \to f$

$$\Delta f = 0, \quad x \in G; \text{ and } f|_{\partial G} = g$$

Variational form
Variational form

Minimize

$$\sum_{x, y : x \simeq y} [u(x) - u(y)]^2$$

over $$u : u = g \in G^c$$
Variational form

Minimize

\[ \sum_{x,y : x \approx y} [u(x) - u(y)]^2 \]

over \( u : u = g \in G^c \)

Converges to the solution that minimizes
Variational form

Minimize

$$\sum_{x,y: x \approx y} [u(x) - u(y)]^2$$

over $$u : u = g \in G^c$$

Converges to the solution that minimizes

$$\int_G |\nabla u|^2 \, dx$$

over $$u : u = g$$ on $$\partial G.$$
Probability theory has an explanation.
Probability theory has an explanation.

Random Walk, \( S_n = \sum_j X_j \) and \( X_j = \pm e_i \) with probability \( \frac{1}{2d} \)
- Probability theory has an explanation.
- Random Walk, $S_n = \sum_j X_j$ and $X_j = \pm e_i$ with probability $\frac{1}{2d}$
- Scales to Brownian Motion.
- Probability theory has an explanation.
- Random Walk, $S_n = \sum_j X_j$ and $X_j = \pm e_i$ with probability $\frac{1}{2d}$
- Scales to Brownian Motion.
- $\sqrt{\frac{d}{n}} S(nt) \to \beta(t)$. 
Probability theory has an explanation.

Random Walk, $S_n = \sum_j X_j$ and $X_j = \pm e_i$ with probability $\frac{1}{2d}$

Scales to Brownian Motion.

$$\sqrt{\frac{d}{n}} S(nt) \to \beta(t).$$

$$E[g(x_\tau)]$$
2. Periodic medium.
2. Periodic medium.

On $\mathbb{R}^d$. $L = \nabla a(x) \cdot \nabla$
2. Periodic medium.

On $\mathbb{R}^d$. $L = \nabla a(x) \cdot \nabla$

$a(x)$ is periodic of period 1 in each $x_i$. 
2. Periodic medium.

On $R^d$. $L = \nabla a(x) \cdot \nabla$

$a(x)$ is periodic of period 1 in each $x_i$.

Interested in scale of size $h^{-1}$ in space and $h^{-2}$ in time.
2. Periodic medium.

On $\mathbb{R}^d$. $L = \nabla a(x) \cdot \nabla$

$a(x)$ is periodic of period 1 in each $x_i$.

Interested in scale of size $h^{-1}$ in space and $h^{-2}$ in time.

$$L_h = \nabla \cdot a\left(\frac{x}{h}\right) \nabla$$
2. Periodic medium.

On $\mathbb{R}^d$. $L = \nabla a(x) \cdot \nabla$

$a(x)$ is periodic of period 1 in each $x_i$.

Interested in scale of size $h^{-1}$ in space and $h^{-2}$ in time.

\[
L_h = \nabla \cdot a\left(\frac{x}{h}\right) \nabla
\]

\[
L_h \rightarrow L = \nabla \cdot \bar{a} \nabla
\]
\[ u_t = L_h u; \quad u(0, x) = f(x) \]
\[ u_t = L_h u; \quad u(0, x) = f(x) \]

\[ L_h u = f \quad \text{for} \quad x \in G; \quad u(y) = g(y) \quad \text{for} \quad y \in \partial G \]
\[ u_t = L_h u; \quad u(0, x) = f(x) \]

\[ L_h u = f \quad \text{for} \quad x \in G; \quad u(y) = g(y) \quad \text{for} \quad y \in \partial G \]

\( \bar{a} \) has a simple variational representation.
\[ u_t = L_h u; \quad u(0, x) = f(x) \]

\[ L_h u = f \quad \text{for} \quad x \in G; \quad u(y) = g(y) \quad \text{for} \quad y \in \partial G \]

\( \bar{a} \) has a simple variational representation.

\[ \langle \xi, \bar{a} \xi \rangle = \inf_w \int_{T^d} \langle (\xi - \nabla w), a(x)(\xi - \nabla w) \rangle \, dx \]
\[ u_t = L_h u; \quad u(0, x) = f(x) \]

\[ L_h u = f \quad \text{for} \quad x \in G; \quad u(y) = g(y) \quad \text{for} \quad y \in \partial G \]

\( \bar{a} \) has a simple variational representation.

\[ \langle \xi, \bar{a} \xi \rangle = \inf_w \int_{T^d} \langle (\xi - \nabla w), a(x)(\xi - \nabla w) \rangle \, dx \]

The inf is taken over periodic functions \( w \).
\[ u_t = L_h u; \quad u(0, x) = f(x) \]

\[ L_h u = f \quad \text{for} \quad x \in G; \quad u(y) = g(y) \quad \text{for} \quad y \in \partial G \]

\( \bar{a} \) has a simple variational representation.

\[ \langle \xi, \bar{a} \xi \rangle = \inf_w \int_{T^d} \langle (\xi - \nabla w), a(x)(\xi - \nabla w) \rangle \, dx \]

The inf is taken over periodic functions \( w \)

In \( d = 1 \), \( \bar{a} = \left[ \int_0^1 \frac{1}{a(x)} \, dx \right]^{-1} \)
3. Random Medium. (Stationary and ergodic)
3. Random Medium. (Stationary and ergodic)

$(\Omega, \mathcal{F}, P), \{T_x\}; x \in \mathbb{R}^d$
3. Random Medium. (Stationary and ergodic)

- $(\Omega, \mathcal{F}, P), \{T_x\}; x \in \mathbb{R}^d$
- $a(x, \omega) = a(T_x \omega)$ is a random positive definite matrix valued function on $\mathbb{R}^d$. 
3. Random Medium. (Stationary and ergodic)

$$(\Omega, \mathcal{F}, P), \{T_x\}; x \in R^d$$

$a(x, \omega) = a(T_x \omega)$ is a random positive definite matrix valued function on $R^d$.

$$L_{h,\omega} = \nabla \cdot a(T_{\frac{x}{h}} \omega) \nabla$$
3. Random Medium. (Stationary and ergodic)

- \((\Omega, \mathcal{F}, P), \{T_x\}; x \in \mathbb{R}^d\)
- \(a(x, \omega) = a(T_x \omega)\) is a random positive definite matrix valued function on \(\mathbb{R}^d\).

\[
L_{h, \omega} = \nabla \cdot a(T_{\frac{x}{h}} \omega) \nabla
\]

\[
\langle \xi, \bar{a} \xi \rangle = \inf_{w^*} E^P [\langle (w^* - \xi), a(\omega)(w^* - \xi) \rangle]
\]
3. Random Medium. (Stationary and ergodic)

- $(\Omega, \mathcal{F}, P), \{T_x\}; x \in \mathbb{R}^d$
- $a(x, \omega) = a(T_x \omega)$ is a random positive definite matrix valued function on $\mathbb{R}^d$.

$$L_{h, \omega} = \nabla \cdot a(T_{\frac{x}{h}} \omega) \nabla$$

$$\langle \xi, \bar{a} \xi \rangle = \inf_{w^*} E^P[\langle (w^* - \xi), a(\omega)(w^* - \xi) \rangle]$$

$$\int w^* dP = 0, D_i w_j^* = D_j w_i^*$$
4. Balanced Case. What if it is periodic but

\[ L_h = \sum_{i,j} a_{i,j} \left( \frac{x}{h} \right) D_i D_j \]
4. Balanced Case. What if it is periodic but

\[ L_h = \sum_{i,j} a_{i,j} \left( \frac{x}{h} \right) D_i D_j \]

\[ \int \phi dx = 1, \phi > 0. \text{ Periodic} \]

\[ L^* \phi = \sum_{i,j} D_i D_j [a_{i,j}(x)\phi(x)] = 0 \]
4. Balanced Case. What if it is periodic but

\[ L_h = \sum_{i,j} a_{i,j} \left( \frac{x}{h} \right) D_i D_j \]

\[ \int \phi dx = 1, \phi > 0. \text{ Periodic} \]

\[ L^* \phi = \sum_{i,j} D_i D_j [a_{i,j}(x) \phi(x)] = 0 \]

\[ \bar{a} = \int_{T^d} a(x) \phi(x) dx \]
4. Balanced Case. What if it is periodic but

\[ L_h = \sum_{i,j} a_{i,j}(\frac{x}{h}) D_i D_j \]

\[ \int \phi dx = 1, \phi > 0. \text{ Periodic} \]

\[ L^* \phi = \sum_{i,j} D_i D_j [a_{i,j}(x)\phi(x)] = 0 \]

\[ \bar{a} = \int_{T^d} a(x)\phi(x) dx \]

In \( d = 1 \) the same answer \( \int [\frac{1}{a(x)} dx]^{-1} \)
Random case.

\[ L_{h,\omega} = \sum_{i,j} a_{i,j}(T_{\frac{x}{h}} \omega) D_i D_j \]

and

\[ \bar{a} = E[a(\omega)\phi(\omega)] \]
Random case.

\[ L_{h,\omega} = \sum_{i,j} a_{i,j}(T_{\frac{x}{h}} \omega) D_i D_j \]

and

\[ \bar{a} = E[a(\omega)\phi(\omega)] \]

\( \phi(\omega) \) is a positive \( L_1 \) function on \((\Omega, \mathcal{F}, P)\)
Random case.

\[ L_{h,\omega} = \sum_{i,j} a_{i,j}(T_{x/h} \omega) D_i D_j \]

and

\[ \bar{a} = E[a(\omega)\phi(\omega)] \]

- \( \phi(\omega) \) is a positive \( L_1 \) function on \((\Omega, \mathcal{F}, P)\)
- It is the unique weak sense solution of

\[ L^* \phi = \sum_{i,j} D_i D_j [a_{i,j}(\omega)\phi(\omega)] = 0 \]
Random case.

\[ L_{h,\omega} = \sum_{i,j} a_{i,j}(T_{\frac{x}{h}}(\omega)) D_i D_j \]

and

\[ \bar{a} = E[a(\omega)\phi(\omega)] \]

\( \phi(\omega) \) is a positive \( L_1 \) function on \((\Omega, \mathcal{F}, P)\)

It is the unique weak sense solution of

\[ L^* \phi = \sum_{i,j} D_i D_j [a_{i,j}(\omega)\phi(\omega)] = 0 \]

\( D_i \) is well defined on \( L_2(P) \)

Hamilton-Jacobi-Bellman equations. $H(x, p)$. Periodic (or Random)

Hamilton-Jacobi-Bellman equations. $H(x, p)$.
Periodic (or Random)

$$u_t + \frac{1}{2} \Delta u + H(x, \nabla u) = 0; \quad u(T, x) = f(\frac{x}{T})$$

Hamilton-Jacobi-Bellman equations. $H(x, p)$. Periodic (or Random)

$u_t + \frac{1}{2} \Delta u + H(x, \nabla u) = 0; \quad u(T, x) = f\left(\frac{x}{T}\right)$

Rescale $(t, x) \rightarrow \left(\frac{t}{T}, \frac{x}{T}\right), \epsilon = T^{-1}$.

$u_t + \frac{\epsilon}{2} \Delta u + H\left(\frac{x}{\epsilon}, \nabla u\right) = 0; \quad u(1, x) = f(x)$

Hamilton-Jacobi-Bellman equations. $H(x, p)$. Periodic (or Random)

$$u_t + \frac{1}{2} \Delta u + H(x, \nabla u) = 0; \quad u(T, x) = f\left(\frac{x}{T}\right)$$

Rescale $(t, x) \rightarrow \left(\frac{t}{T}, \frac{x}{T}\right)$, $\epsilon = T^{-1}$.

$$u_t + \frac{\epsilon}{2} \Delta u + H\left(\frac{x}{\epsilon}, \nabla u\right) = 0; \quad u(1, x) = f(x)$$

$$u_t + \bar{H}(\nabla u) = 0; \quad u(1, x) = f(x)$$
How is $\bar{H}(\rho)$ related to $H(x, \rho)$?
How is $\bar{H}(p)$ related to $H(x, p)$?

$L(x, q)$ is the Legendre transform.
How is $\bar{H}(p)$ related to $H(x, p)$?

$L(x, q)$ is the Legendre transform.

Consider $A_b = \frac{1}{2} \Delta + \langle b(x), \nabla \rangle$ on the torus.
How is $\bar{H}(p)$ related to $H(x, p)$?

$L(x, q)$ is the Legendre transform.

Consider $A_b = \frac{1}{2} \Delta + \langle b(x), \nabla \rangle$ on the torus.

$A_b^* \phi_b = 0$
How is $\bar{H}(p)$ related to $H(x, p)$?

$L(x, q)$ is the Legendre transform.

Consider $A_b = \frac{1}{2} \Delta + < b(x), \nabla >$ on the torus.

$A_b^* \phi_b = 0$

$\bar{H}(p) =$

$$\sup_{b(\cdot)} \left[ < p, \int b(x) \phi_b(x) > - \int L(x, b(x)) \phi_b(x) \right]$$
In the random case $\phi_b$ does not always exist.
In the random case $\phi_b$ does not always exist.

Limit the variational formula to $b$ such that $\phi_b$ exists.
In the random case $\phi_b$ does not always exist.

Limit the variational formula to $b$ such that $\phi_b$ exists.

Goes under the general name of "homogenization"
In the random case $\phi_b$ does not always exist.

Limit the variational formula to $b$ such that $\phi_b$ exists.

Goes under the general name of "homogenization"

7. Interacting particle systems.
7. Interacting particle systems.

On $\mathbb{Z}^d$ or $\mathbb{R}^d$ we have particles.
7. Interacting particle systems.

On $\mathbb{Z}^d$ or $\mathbb{R}^d$ we have particles.

They interact with each other and move. From a distance (rescaled) it looks a cloud of particles moving.
7. Interacting particle systems.

On $Z^d$ or $R^d$ we have particles.

They interact with each other and move. From a distance (rescaled) it looks a cloud of particles moving.

The density at time $t$ is a some $\rho(t, x)$. 
7. Interacting particle systems.

On $Z^d$ or $R^d$ we have particles.

They interact with each other and move. From a distance (rescaled) it looks a cloud of particles moving.

The density at time $t$ is some $\rho(t, x)$.

How does it evolve?
Simple exclusion process.
Simple exclusion process.

\[ \mathcal{L} f = \sum_{x, y} \eta(x)(1 - \eta(y))p(y - x)[f(\eta^x, y) - f(\eta)] \]
Simple exclusion process.

\[ \mathcal{L} f = \sum_{x,y} \eta(x)(1 - \eta(y))p(y - x)[f(\eta^{x,y}) - f(\eta)] \]

Invariant distributions are Bernoulli.
Simple exclusion process.

\[ \mathcal{L}f = \sum_{x,y} \eta(x)(1 - \eta(y))p(y - x)[f(\eta^{x,y}) - f(\eta)] \]

Invariant distributions are Bernoulli.

Rescale \( x \rightarrow Nx, \ t \rightarrow N^2t \).
Simple exclusion process.

\[ \mathcal{L} f = \sum_{x,y} \eta(x)(1 - \eta(y))p(y - x)[f(\eta^{x,y}) - f(\eta)] \]

Invariant distributions are Bernoulli.

Rescale \( x \rightarrow Nx, \ t \rightarrow N^2t. \)

Start far away from equilibrium.
Simple exclusion process.

\[ \mathcal{L} f = \sum_{x,y} \eta(x)(1 - \eta(y))p(y - x)[f(\eta^{x,y}) - f(\eta)] \]

Invariant distributions are Bernoulli.

Rescale \( x \rightarrow Nx, \ t \rightarrow N^2t \).

Start far away from equilibrium.

How does the density evolve to equilibrium?
Look at

\[ F_J(\eta) = \frac{1}{N^d} \sum J\left(\frac{x}{N}\right)\eta(x) \]
Look at

\[ F_J(\eta) = \frac{1}{N^d} \sum J\left(\frac{x}{N}\right)\eta(x) \]

Compute \( N^k \mathcal{L} F_J \)
Look at

\[ F_J(\eta) = \frac{1}{N^d} \sum J\left(\frac{x}{N}\right)\eta(x) \]

Compute \( N^k \mathcal{L} F_J \)

\( k = 1 \) if \( \sum z p(z) = m \neq 0 \)
Look at

\[ F_J(\eta) = \frac{1}{N^d} \sum J\left(\frac{x}{N}\right)\eta(x) \]

Compute \( N^k \mathcal{L} F_J \)

- \( k = 1 \) if \( \sum z p(z) = m \neq 0 \)
- \( k = 2 \) if \( \sum z p(z) = 0 \)
Look at

\[ F_J(\eta) = \frac{1}{N^d} \sum J(\frac{x}{N})\eta(x) \]

Compute \( N^k \mathcal{L} F_J \)

- \( k = 1 \) if \( \sum_z zp(z) = m \neq 0 \)
- \( k = 2 \) if \( \sum_z zp(z) = 0 \)

If \( p(z) = p(-z) \) it is a lot easier.
\[ N^{2-d} \sum \eta(x)(1 - \eta(y))p(y-x)[f(\eta^{x,y}) - f(\eta)] \]

\[ = \frac{N^{2-d}}{2} \sum_{x,y} [\eta(x) - \eta(y)]p(y-x)J[(\frac{y}{N}) - J(\frac{x}{N})] \]

\[ \sim \frac{1}{2N^d} \sum_x C_{i,j}(\partial_i \partial_j f)(\frac{x}{N}) \]
\[ N^{2-d} \sum \eta(x)(1 - \eta(y))p(y - x)[f(\eta^{x,y}) - f(\eta)] \]

\[ = \frac{N^{2-d}}{2} \sum_{x,y} [\eta(x) - \eta(y)]p(y - x)J[(\frac{y}{N}) - J(\frac{x}{N})] \]

\[ \approx \frac{1}{2N^d} \sum_x C_{i,j}(\partial_i \partial_j f)(\frac{x}{N}) \]

where \( C_{i,j} = \sum_z z_i z_j p(z) \)
The evolution of density is given by

\[ \rho_t = \frac{1}{2} \nabla \cdot C \nabla \rho \]
The evolution of density is given by

\[ \rho_t = \frac{1}{2} \nabla \cdot C \nabla \rho \]

Interaction does not seem to play a role.
The evolution of density is given by

\[ \rho_t = \frac{1}{2} \nabla \cdot C \nabla \rho \]

Interaction does not seem to play a role.

But if \( p(z) \) is replaced by \( p(z) + \frac{q(z)}{N} \) with \( \sum_z zq(z) = m \)
The evolution of density is given by

\[ \rho_t = \frac{1}{2} \nabla \cdot C \nabla \rho \]

Interaction does not seem to play a role.

But if \( p(z) \) is replaced by \( p(z) + \frac{q(z)}{N} \) with \( \sum_z zq(z) = m \)

\[ \rho_t = \frac{1}{2} \nabla \cdot C \nabla \rho - \nabla \cdot m \rho (1 - \rho) \]
The evolution of density is given by

\[ \rho_t = \frac{1}{2} \nabla \cdot C \nabla \rho \]

Interaction does not seem to play a role.

But if \( p(z) \) is replaced by \( p(z) + \frac{q(z)}{N} \) with \( \sum_z z q(z) = m \)

\[ \rho_t = \frac{1}{2} \nabla \cdot C \nabla \rho - \nabla \cdot m \rho (1 - \rho) \]

The average of \( \eta(x)(1 - \eta(y)) \) is replaced by its local expectation \( \rho(1 - \rho) \).
If $\sum z^p(z) = m \neq 0$ then with $x \to Nx$ and $t \to Nt$. 
If \( \sum_z z \rho(z) = m \neq 0 \) then with \( x \to Nx \) and \( t \to Nt \),

\[
\rho_t + \nabla \cdot m \rho (1 - \rho) = 0
\]
If $\sum z p(z) = m \neq 0$ then with $x \to Nx$ and $t \to Nt$,

$$\rho_t + \nabla \cdot m \rho (1 - \rho) = 0$$

If $p(z)$ is not symmetric but $\sum z p(z) = 0$, 

$$\sum z p(z) = m \neq 0$$
If $\sum_z z p(z) = m \neq 0$ then with $x \to N x$ and $t \to N t$,

$$\rho_t + \nabla \cdot m \rho (1 - \rho) = 0$$

If $p(z)$ is not symmetric but $\sum_z p(z) = 0$,

$t \to N^2 t$, $C$ now depends on $\rho$
If \( \sum_z z p(z) = m \neq 0 \) then with \( x \to N x \) and \( t \to N t \),

\[
\rho_t + \nabla \cdot m \rho (1 - \rho) = 0
\]

If \( p(z) \) is not symmetric but \( \sum_z p(z) = 0 \),

\( t \to N^2 t \), \( C \) now depends on \( \rho \)

\[
\rho_t = \frac{1}{2} \nabla \cdot C'(\rho) \nabla \rho
\]
\begin{itemize}
  \item If \( \sum_z z p(z) = m \neq 0 \) then with \( x \to Nx \) and \( t \to Nt \),
  \[ \rho_t + \nabla \cdot m \rho (1 - \rho) = 0 \]
  \item If \( p(z) \) is not symmetric but \( \sum_z p(z) = 0 \),
  \item \( t \to N^2 t \), \( C \) now depends on \( \rho \)
  \item \( \rho_t = \frac{1}{2} \nabla \cdot C(\rho) \nabla \rho \)
  \item \( C \) is not easily computable.
\end{itemize}
Generator

\[ \mathcal{L}_N = N^2 \sum_{x,y} p(y-x) \eta(x)(1-\eta(y)) [F(\eta^{x,y}) - F(\eta)] \]
Generator

\[ \mathcal{L}_N = N^2 \sum_{x,y} p(y-x) \eta(x)(1-\eta(y))[F(\eta^{x,y}) - F(\eta)] \]

\[ F_J(\eta) = \langle J, \rho \rangle = \frac{1}{N^d} \sum_x J\left(\frac{x}{N}\right)\eta(x) \]
\[ \mathcal{L}_N \mathcal{F}_J = N^{2-d} \sum_{x,y} p(y - x) \eta(x)(1 - \eta(y)) \]

\[ \times \left[ J\left( \frac{y}{N} \right) - J\left( \frac{x}{N} \right) \right] \]
\[ \mathcal{LF}_J = N^{2-d} \sum_{x,y} p(y - x) \eta(x)(1 - \eta(y)) \times [J(\frac{y}{N}) - J(\frac{x}{N})] \]

\[ \approx N^{1-d} \sum_x \nabla J(\frac{x}{N}) \cdot W(x, \eta) \]
if $p(\cdot)$ is symmetric

$$W_i(x, \eta) = \frac{1}{2} \sum_j C_{i,j} [\eta(x + e_j) - \eta(x)]$$

Can do summation by parts.
if \( p(\cdot) \) is symmetric

\[
W_i(x, \eta) = \frac{1}{2} \sum_j C_{i,j} [\eta(x + e_j) - \eta(x)]
\]

Can do summation by parts.

Otherwise use \( E[W] = 0 \) in every equilibrium.

\[
W_i(x, \eta) \approx \frac{1}{2} \sum_j C_{i,j} [\eta(x + e_j) - \eta(x)] + ??
\]
if $p(\cdot)$ is symmetric

$$W_i(x, \eta) = \frac{1}{2} \sum_j C_{i,j} [\eta(x + e_j) - \eta(x)]$$

Can do summation by parts.

Otherwise use $E[W] = 0$ in every equilibrium.

$$W_i(x, \eta) \approx \frac{1}{2} \sum_j C_{i,j} [\eta(x + e_j) - \eta(x)] + ??$$

The ?? can be ignored
$E[W_i] = 0$ in all equilibria form a Hilbert space.
\( E[W_i] = 0 \) in all equilibria form a Hilbert space.

There is subspace that are negligible.
- $E[W_i] = 0$ in all equilibria form a Hilbert space.
- There is subspace that are negligible.
- Codimension $d$ in the Hilbert space.
\( E[W_i] = 0 \) in all equilibria form a Hilbert space.

There is subspace that are negligible.

Codimension \( d \) in the Hilbert space.

Density gradients \( \eta(x + e_j) - \eta(x) \) are complementary.
- $E[W_i] = 0$ in all equilibria form a Hilbert space.
- There is subspace that are negligible.
- Codimension $d$ in the Hilbert space.
- Density gradients $\eta(x + e_j) - \eta(x)$ are complementary.
- Done in each equilibrium $P_\rho$ with $C_{i,j}(\rho)$
- $E[W_i] = 0$ in all equilibria form a Hilbert space.
- There is subspace that are negligible.
- Codimension $d$ in the Hilbert space.
- Density gradients $\eta(x + e_j) - \eta(x)$ are complementary.
- Done in each equilibrium $P_\rho$ with $C_{i,j}(\rho)$
- Large Deviation theory.

What about the motion of a tagged particle in equilibrium at density $\rho$?

What about the motion of a tagged particle in equilibrium at density $\rho$?

Diffuses. $\lambda^{-1} x(\lambda^2 t) \to B(t)$. 

What about the motion of a tagged particle in equilibrium at density $\rho$?

Diffuses. $\lambda^{-1} x(\lambda^2 t) \rightarrow B(t)$.

Covariance is $S(\rho)$.
What about in non-equilibrium?
What about in non-equilibrium?

\[ L_t = \frac{1}{2} \nabla \cdot S(\rho(t, x)) \nabla + \frac{(S(\rho(t, x)) - C') \nabla \rho}{2\rho} \]
What about in non-equilibrium?

\[ L_t = \frac{1}{2} \nabla \cdot S(\rho(t, x)) \nabla + \frac{(S(\rho(t, x)) - C') \nabla \rho}{2\rho} \nabla \]

\[ L_t^* \rho = \frac{1}{2} \nabla C \nabla \rho \]
What about in non-equilibrium?

\[ L_t = \frac{1}{2} \nabla \cdot S(\rho(t, x)) \nabla + \frac{(S(\rho(t, x)) - C)}{2\rho} \nabla \rho \nabla \]

\[ L^*_t \rho = \frac{1}{2} \nabla C \nabla \rho \]

9. Trajectories
What about in non-equilibrium?

\[ L_t = \frac{1}{2} \nabla \cdot S(\rho(t, x)) \nabla + \frac{(S(\rho(t, x)) - C) \nabla \rho}{2\rho} \nabla \]

\[ L^*_t \rho = \frac{1}{2} \nabla C \nabla \rho \]

9. Trajectories

\[ \frac{1}{N^d} \sum_i \delta x_i (N^2.) \rightarrow P \]
What about in non-equilibrium?

\[ L_t = \frac{1}{2} \nabla \cdot S(\rho(t, x)) \nabla + \frac{(S(\rho(t, x)) - C)}{2\rho} \nabla \rho \nabla \]

\[ L^*_t \rho = \frac{1}{2} \nabla C \nabla \rho \]

9. Trajectories

\[ \frac{1}{N^d} \sum_i \delta_{x_i(\frac{N^2}{N})} \rightarrow P \]

Markov with generator \( L_t \).

ODE’s

\[
\begin{align*}
\dot{q}_i &= p_i, \\
\dot{p}_i &= -\sum_j (\nabla V)(x_i - x_j)
\end{align*}
\]

ODE’s

\[ \dot{q}_i = p_i, \quad \dot{p}_i = - \sum_j (\nabla V)(x_i - x_j) \]

Conserved quantities. \( \rho, u, e. \)

ODE’s

\[ \dot{q}_i = p_i, \quad \dot{p}_i = - \sum_j (\nabla V)(x_i - x_j) \]

Conserved quantities. \( \rho, u, e \).

First order hyperbolic PDE for them.

ODE’s

\[ \dot{q}_i = p_i, \quad \dot{p}_i = - \sum_j (\nabla V)(x_i - x_j) \]

Conserved quantities. \( \rho, u, e \).

First order hyperbolic PDE for them.

Connect the ODE’s to PDE
Gibbs States. Constant $\rho, u, e.$
- Gibbs States. Constant $\rho, u, e$.
- Local Gibbs state, Slowly varying $\rho, u, e$
- Gibbs States. Constant $\rho, u, e$.
- Local Gibbs state, Slowly varying $\rho, u, e$
- Liouville flow
Gibbs States. Constant $\rho, u, e$.

Local Gibbs state, Slowly varying $\rho, u, e$

Liouville flow

Euler Flow.
Gibbs States. Constant $\rho, u, e$.

Local Gibbs state, Slowly varying $\rho, u, e$

Liouville flow

Euler Flow.

\[
\begin{align*}
[\rho_0, u_0, T_0] & \rightarrow \text{Local Gibbs} \\
\downarrow \text{Euler} & \quad \downarrow \text{Liouville} \\
[\rho_t, u_t, T_t] & \rightarrow \text{do not match}
\end{align*}
\]
Gibbs States. Constant $\rho, u, e$.

Local Gibbs state, Slowly varying $\rho, u, e$

Liouville flow

Euler Flow.

\[
\begin{align*}
[\rho_0, u_0, T_0] & \quad \longrightarrow \quad \text{Local Gibbs} \\
& \quad \downarrow \quad \text{Euler} \\
& \quad \quad \quad \quad \quad \quad \rightarrow \\
[\rho_t, u_t, T_t] & \quad \longrightarrow \quad \text{do not match}
\end{align*}
\]

Diagram does not commute!
- Gibbs States. Constant $\rho, u, e$.
- Local Gibbs state, Slowly varying $\rho, u, e$
- Liouville flow
- Euler Flow.

$$[\rho_0, u_0, T_0] \rightarrow \text{Local Gibbs}$$

$$\downarrow \text{Euler} \hspace{1cm} \downarrow \text{Liouville}$$

$$[\rho_t, u_t, T_t] \rightarrow \text{do not match}$$

- Diagram does not commute!
- It almost does after some noisy modification.
Work done over 25 years. Presutti, De Masi, H.T. Yau, Olla, Rezakhanlou, Quastel, Kosygina, V
Work done over 25 years. Presutti, De Masi, H.T.Yau, Olla, Rezakhanlou, Quastel, Kosygina, V
Thank You