Scaling Limits

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- We will look at some examples.

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- Courant-Friedrichs-Lewy 1928. Math. Ann.

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$$\int_G |\nabla u|^2 dx$$

over $u : u = g$ on ∂G.

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- $E[g(x_\tau)]$

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- In $d = 1$, $\bar{a} = [\int_0^1 \frac{1}{a(x)} dx]^{-1}$

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- $\int w^* dP = 0, D_i w_j^* = D_j w_i^*$

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- In $d = 1$ the same answer $\int [\frac{1}{a(x)} dx]^{-1}$

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- D_i is well defined on $L_2(P)$

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$$\sup_{b(\cdot)} [< p, \int b(x)\phi_b(x) > - \int L(x, b(x))\phi_b(x)]$$

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- J.L.Lions, P.L.Lions, A.Bensoussan, G.C.Papanicolaou, F.Rezakhanlou, E.Kosygina, P.E. Suganidis & V.

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- How does it evolve?

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- Start far away from equilibrium.
- How does the density evolve to equilibrium?

- Look at

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- $k = 2$ if $\sum_z zp(z) = 0$
- If $p(z) = p(-z)$ it is a lot easier.

$$N^{2-d}\sum \eta(x)(1-\eta(y))p(y-x)[f(\eta^{x,y})-f(\eta)]$$

$$= \frac{N^{2-d}}{2}\sum_{x,y}[\eta(x)-\eta(y)]p(y-x)J[(\frac{y}{N})-J(\frac{x}{N})]$$

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- where $C_{i,j}=\sum_z z_i z_j p(z)$

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- $$\rho_t = \frac{1}{2}\nabla \cdot C\nabla \rho - \nabla \cdot m\rho(1-\rho)$$
- The average of $\eta(x)(1-\eta(y))$ is replaced by its local expectation $\rho(1-\rho)$.

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- C is not easily computable.

- Generator

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$$F_J(\eta) =< J, \rho >= \frac{1}{N^d} \sum_x J(\frac{x}{N})\eta(x)$$

$$\mathcal{L}_N \mathcal{F}_{\mathcal{J}} = N^{2-d} \sum_{x,y} p(y-x)\eta(x)(1-\eta(y))$$
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- $$\simeq N^{1-d} \sum_x \nabla J\left(\frac{x}{N}\right) \cdot W(x,\eta)$$

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- The ?? can be ignored

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- It almost does after some noisy modification.

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Thank You