Preliminary Exam: Probability, August 2024.
Modality: In-person.
Time: 10:00am - 3:00pm, Friday, August 23, 2024.
Place: C506 Wells Hall.

Your goal should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of 6 main problems, each with several steps designed to help you in the overall solution.

Important: **If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!**

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.

2. Start each problem on a new page.

Problem 1. Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables such that $E(X_n) = 0$,

$$E(X_n^2) = \frac{1}{n\ln(n+1)},\ E(X_n X_{n+1}) = \frac{2}{n^2},\ \text{ and }\ E(X_m X_n) = 0 \ \text{ if } \ |m-n| \geq 2.$$

a. Denote by σ_n^2 the variance of $S_n = \sum_{i=1}^{n} X_i$. Prove that $\lim\limits_{n\to\infty} \frac{\sigma_n^2}{\ln\ln n} = 1$.

b. Prove that for every sequence $\{a_n,\ n \geq 1\}$ of positive numbers with $\lim\limits_{n\to\infty} a_n = \infty$, we have

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{a_n \cdot \ln\ln n}} = 0 \text{ in probability and in } L^2(\mathrm{P}).$$

c. Prove that for every $\varepsilon > 0$ the following holds:

(i) $\sum_{n=1}^{\infty} E\left(\frac{S_n^2}{n \cdot \ln n \cdot (\ln\ln n)^{2+\varepsilon}}\right) < \infty$, and $\sum_{n=1}^{\infty} \frac{S_n^2}{n \cdot \ln n \cdot (\ln\ln n)^{2+\varepsilon}}$ converges a.s.

(ii) $\lim\limits_{n\to\infty} \frac{S_n}{\sqrt{n \cdot \ln n \cdot (\ln\ln n)^{2+\varepsilon}}} = 0 \quad$ a.s.

Problem 2. Let $X, X_1, X_2, \ldots$ be i.i.d. sequence of positive random variables. Let $0 < \beta < 1$. Assume

$$P(X > x) \leq x^{-\beta}, x > 1$$

Let $\{a_n\}_{n=1,2,\ldots}$ be a sequence of positive real numbers that satisfies: $\sum_{n=1}^{\infty} a_n^{\beta} < \infty$. Prove the following:

a. $\sum_{n=1}^{\infty} P(a_n X > 1) < \infty$.

b. (i) $\sum_{n=1}^{\infty} E(a_n X \cdot 1_{\{a_n X < 1\}}) < \infty$,

(ii) $\sum_{n=1}^{\infty} E(a_n^2 X^2 \cdot 1_{\{a_n X < 1\}}) < \infty$.

c. (i) $\sum_{n=1}^{\infty} a_n X_n < \infty$, a.s.

(ii) Assume also that $\{a_n\}_{n=1,2,\ldots}$ is non-decreasing. Prove: $a_n \cdot \sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{} 0$, a.s.

Problem 3. Let $\{X,\ X_n,\ n\ \geq 1\}$ be a sequence of i.i.d. random variables whose characteristic function satisfies

$$\varphi(t) = e^{-|t|^{\alpha}(1\ +|t|)}\ for\ -1 < t < 1,$$

where $\alpha\ \in\ (0, 2]$ is a constant. For $n \geq 1$, let $S_n\ =\ \sum_{i=1}^{n} X_i$ and let $\varphi_n(t)$ be the characteristic functions of $n^{-1/\alpha}S_n$.

a. Find $\lim_{n \to \infty} \varphi_n(t)$.

b. Prove that $n^{-1/\alpha}S_n$ converge in distribution to a random variable *Y*.

c. Prove that the random variable *Y* in (ii) has a continuous and bounded probability density function.

Problem 4. Let $\{(X_{k,}Y_k)\}_{k=1,2,\ldots}$ be a sequence of pairs of random variables. Denote

$$S_n = \sum_{k=1}^{n} X_k,\ T_n = \sum_{k=1}^{n} Y_k,\ n = 1, 2, \ldots$$

a. Assume that $\sum_{k=1}^{\infty} P(X_k \neq Y_k) < \infty$, and let $a_n \xrightarrow[n\to\infty]{} \infty$. Prove that if $\frac{T_n}{a_n}$ converges in distribution to W, then $\frac{S_n}{a_n}$ converges in distribution to W as well.

b. From now on assume that $\{\ X_k\}_{k=1,2,\ldots}$ are independent and $X_1 = 0$,

$$X_k = \begin{cases} \pm 1 \text{ with probability } \frac{1}{2} - \frac{1}{2\cdot k^2} \\ \pm k \text{ with probability } \frac{1}{2\cdot k^2} \end{cases},\ k = 2, 3, \ldots$$

Let $Y_0 = 0,\ Y_k = X_k \cdot 1_{\{X_k = \pm 1\}}, k = 1,2, \ldots$

Prove that $\frac{Var(S_n)}{2n} \xrightarrow[n\to\infty]{} 1$ and $\frac{Var(T_n)}{n} \xrightarrow[n\to\infty]{} 1.$

c. (i) Does the triangular array $\{\frac{X_k}{\sqrt{2n}}\}_{k=1,\ldots,n,\ \ n=1,2,\ldots}$ satisfy Lindberg condition? What about the triangular array $\{\frac{Y_k}{\sqrt{n}}\}_{k=1,\ldots,n,\ \ n=1,2,\ldots}$?

(ii) Prove that $\frac{S_n}{\sqrt{n}}$ converges in distribution to $N(0, 1)$.

Problem 5. Here $n = 1, 2, \ldots$ Let $\{\mathcal{F}_n\}$ be a sequence of σ-algebras that satisfies $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Let $T < \infty$, a.s. be a stopping time with respect to $\{\mathcal{F}_n\}$. Let $\{X_n\}$ be a sequence of random variables, so that $X_n \in \mathcal{F}_n$, and $E(X_n) = E(X_1)$. Finally, assume that $\sigma(X_{n+1})$and $\mathcal{F}_n$ are independent.

a. Let $S_n = \sum_{k=1}^{n} X_k$, $S_0 = 0$ Prove:
   (i) $S_{T\wedge(n+1)} - S_{T\wedge n}$=$X_{n+1} \cdot 1_{\{T\geq n+1\}}$, where $a \wedge b = \min\{a, b\}, a, b \in \mathcal{R}$
   (ii) $S_T = \sum_{n=0}^{\infty} S_{T\wedge(n+1)} - S_{T\wedge n}$.
b. Assume in this part that $X_n \geq 0$, a.s. Prove:
   (i) Show by using part a that $E(S_T) = E(X_1) \cdot E(T)$. Observe that $E(T)$ can be either finite or infinite.
   Hint: start by showing that $E\left(S_{T\wedge(n+1)} - S_{T\wedge n}\right) = E(X_1) \cdot P(T \geq n+1)$
   (ii) As an example, consider $\{X_n\}$ to be i.i.d. and $P(X_1 = 0) = P(X_1 = 1) = 1/2$. Let $T = \min_{n\geq 1}\{S_n = 2\}$. What is $E(T)$? Also, how is T distributed?
c. We drop here the assumption $X_n \geq 0$. Assume instead that $\sup_{n\geq 1} E(|X_n|) < \infty$, and that $E(T) < \infty$. Prove:
   (i) E($\sum_{n=0}^{\infty} |S_{T\wedge(n+1)} - S_{T\wedge n}|$)<∞
   (ii) Use c(i) to prove that $E(S_T) = E(X_1) \cdot E(T)$ still holds.
   Hint: Show first that $S_{T\wedge m} \xrightarrow[m\to\infty]{} S_T$, a.s., and then use a(ii) for the stopping time $T \wedge m$ which is bounded by m.

Problem 6. Let $\{B_1(t), B_2(t), t \geq 0\}$ be 2 independent standard Brownian motions (SBM). Let $\Omega = \{(t, u) \in R^2\colon\ 0 \leq t, u \leq 1\}$, and let $H = L^2(\Omega, \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ algebra, and λ is the Lebesgue measure. Let $\langle f, g\rangle = \int_{u=0}^{1}\int_{t=0}^{1} f(t, u) \cdot g(t, u)\mathrm{dtdu}, f, g \in H$.

In what follows $(t, u) \in \Omega$.

a. Let $B(t, u) = B_1(t) \cdot B_2(u)$. Calculate:
   (i) E $(B(t, u))$, and
   (ii) $\mathrm{COV}[B(t_1, u_1), B(t_2, u_2)]$.
b. Let $\{\varphi_k\}_{k\geq 1} \subseteq H$ be a complete orthonormal basis of H, namely $\langle\varphi_k, \varphi_m\rangle = \begin{cases} 1 & if\ k = m \\ 0 & if\ k \neq m \end{cases}$, $f = \sum_{k=1}^{\infty}\langle f, \varphi_k\rangle\, \varphi_k$, and $\langle f, g\rangle = \sum_{k=1}^{\infty}\langle f, \varphi_k\rangle\langle g, \varphi_k\rangle$, for every $f, g \in H$.
   Let $\{Z_k\}_{k\geq 1}$ be independent and standard normal random variables.
   Define $T(f) = \sum_{k=1}^{\infty}\langle f, \varphi_k\rangle\, Z_k, f \in H$.
   (i) Verify that the series $\sum_{k=1}^{\infty}\langle f, \varphi_k\rangle\, Z_k$ converges almost surely. Thus, $T(f)$ is well defined for every $f \in H$.
   (ii) Prove that $\langle f, g\rangle = \mathrm{COV}[T(f), T(g)]$ for all $f, g \in H$.
   (iii) How is $T(f)$ distributed?
c. Prove for $(t, u) \in \Omega$ and $f_{t,u}(x, y) = \begin{cases} 1 & if\ 0 \leq x \leq t, 0 \leq y \leq u \\ 0 & \text{otherwise} \end{cases}$
   (i) $E(B(t, u)) = E\left(T\left(f_{t,u}\right)\right)$, and $\mathrm{COV}[B(t_1, u_1), B(t_2, u_2)] = \mathrm{COV}\left[T\left(f_{t_1,u_1}\right), T\left(f_{t_2,u_2}\right)\right]$
   (ii) Is $B(t, u) = T\left(f_{t,u}\right)$ in distribution? Explain.