Preliminary Exam: Probability.

Time: 10:00am - 3:00pm, Thursday, August 17, 2023.
Place: C506 Wells Hall.

Your goal should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: **If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!**

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.

2. Start each problem on a new page.

Problem 1. Let $\{B_k\}_{k=1,2,3,4}$ be 4 events in a probability space.

Let $1_B(\omega) = \begin{cases} 1, & \omega \in B \\ 0, & \omega \in B^c \end{cases}$ be the indicator function of any event B whose complement is B^c.

Let $A = \bigcap_{k=1}^{3} B_k$.

a. Prove $1_A = \prod_{k=1}^{3}(1 - 1_{B_k^c})$

b. Show by using part a that

$$P(A) = 1 - \sum_{1\le k\le 3} P(B_k^c) + \sum_{1\le k<l\le 3} P(B_k^c \cap B_l^c) - P(\bigcap_{k=1}^{3} B_k^c)$$

c. Extend part b to the case $A = \bigcap_{k=1}^{4} B_k$.

Problem 2. Let $\{X_k\}$, $k = 1, 2, \ldots$ be a sequence of random variables. In this problem all convergences are as $k \to \infty$.

a. Prove that $X_k \to 0$ in probability if and only if $X_k \to 0$ in distribution.

b. (i) Prove that if $X_k \to 0$, a.s. then $P(\cup_{m=k}^{\infty}\{|X_m| > \varepsilon\}) \to 0$ for each $\varepsilon > 0$.
(ii) Show how to conclude from (i) that convergence a.s. implies convergence in probability.

c. Assume that $\{X_k\}$ are independent, P$(|X_k| > 1) = \frac{1}{k}$, $k = 1, 2, \ldots$ and P$(|X_k| \leq \frac{1}{\sqrt{k}}) \to 1$. Show that $X_k \to 0$ in probability, **but** $X_k \to 0$ a.s. is false.

Problem 3. Let $X, X_1, \dots, X_n, \dots$ be a sequence of i.i.d. random variables such that X is symmetric (namely $X = -X$ in distribution), and $P(|X| > x) = \begin{cases} 1 \text{ if } 0 \le x < e \\ \frac{1}{x^2 \cdot \ln(x)} \text{ if } x \ge e \end{cases}$

Let $Y_{n,m} \equiv X_m \cdot 1_{\{|X_m| \le \sqrt{n}\}}$, $m = 1, \dots, n,\ n = 1,2, \dots$

Solve the following:

a. Calculate $E(X^2)$.

b. Prove

(i) $\sum_{m=1}^{n} P(Y_{n,m} \ne X_m) \xrightarrow[n\to\infty]{} 0.$

(ii) $E(Y_{n,m}^2) \sim 2\ln(\ln(n))$, as $n \to \infty$.

c. Prove that the following two sequences converge in distribution as $n \to \infty$, and identify the limit distribution.

(i) $\frac{\sum_{m=1}^{n} Y_{n,m}}{\sqrt{2\ln(\ln(n))}}$

(ii) $\frac{\sum_{m=1}^{n} X_m}{\sqrt{2\ln(\ln(n))}}$

Problem 4. Let $\{B_t, t \geq 0\}$ be a standard Brownian motion (SBM) equipped with its canonical filtration, $\{\mathcal{F}_t\}, t \geq 0$.

Let X be a random variable which gets exactly 3 distinct and non-zero values denoted by $\{x_k\}$, $k = 1, 2, 3$. Also, let $A_k = \{X = x_k\}$, $P(A_k) = p_k > 0, k = 1, 2, 3$. Assume that $E(X) = 0$.

a. Let $\mathcal{G} = \sigma\{A_1, A_2 \cup A_3\}$ and denote $Y = E_{\mathcal{G}}(X)$.
   (i). How many values does Y get? What is $E(Y)$? What can you say about the event $\{X = Y\}$?
   (ii) Use the symbols $\{x_k, p_k\}, k = 1, 2, 3$ to present the explicit distribution of Y.

b. Define a stopping time with respect to the SBM filtration, denoted by τ_1, so we have $Y = B_{\tau_1}$ in distribution. Prove your answer.

c. Find a stopping time with respect to the SBM filtration, denoted by τ_2, so that
   (i) $X = B_{\tau_2}$ in distribution, and
   (ii) $\tau_2 \geq \tau_1, \mathrm{a.s.}$

   Hint. Differentiate between the events $\{X = Y\}$ and $\{X \neq Y\}$. Also, let $Z_t = B_{\tau_1 + t}, t \geq 0$. By the strong Markov property $Z_t - B_{\tau_1}$ $t \geq 0$, is a SBM which is independent of $\mathcal{F}_{\tau_1}$.

Problem 5. Let $\{\varepsilon, \varepsilon_k : k = 1, 2, \ldots\}$ be i.i.d. sequence of random variables, with

$P(\varepsilon = \pm 1) = 1/2$.

Let $S_n = \sum_{k=1}^{n} \varepsilon_k$, $n = 1, 2, \ldots, S_0 = 0$ (so $\{S_n\}$ is a simple symmetric random walk.)

Let $\{\mathcal{F}_n : n = 0, 1, \ldots \}$ be the natural filtration of $\{S_n\}$.

Finally, let $\tau = \min\{n : S_n = 1\}$.

a. (i) Prove that τ is a stopping time with respect to $\{\mathcal{F}_n\}$.

   (ii) Let $\{H_n : n = 1, \ldots \}$ be a bounded sequence of random variables so that $H_n \in \mathcal{F}_{n-1}, n = 1, 2, \ldots$ Prove that $\{\sum_{k=1}^{n} H_k \varepsilon_k , \mathcal{F}_n \}$ is a martingale.

b. (i) Prove that $\{S_{n\wedge\tau}, \mathcal{F}_n \}$ is a martingale. Do it by using part a(ii) with an appropriate choice of $\{H_n\}$.

   (ii) Prove that $S_{n\wedge\tau}$ converges a.s. as $n \to \infty$, a.s.

c. (i) Identify the distribution of the limit of the sequence $S_{n\wedge\tau}$ as $n \to \infty$.

   (ii) Prove that $\{S_{n\wedge\tau}\}$ isn't uniformly integrable.

Problem 6. Let $\{\varepsilon_k\}$ be independent sequence of random variables, with

$E(\varepsilon_k) = 0$ and $E(\varepsilon_k^2) < \infty, k = 1, 2, \ldots$ Let $S_n = \sum_{k=1}^{n} \varepsilon_k$, and let $\{\mathcal{F}_n\}$ be the natural filtration of $\{S_n\}$, $n = 1, 2, \ldots$ Let $A = \{\max_{1 \le n \le m} |S_n| > x\}$, where $x > 0$, and the integer $m \ge 1$ are fixed. Let $\tau = \min\{n \le m : |S_n| > x \text{ or } n = m\}$.

a. Prove the following
   (i) $(S_n^2 - \sigma_n^2, \mathcal{F}_n)$ is a martingale, where $\sigma_n^2 = E(S_n^2)$.

   (ii) $E\left(S_\tau^2 - \sigma_\tau^2\right) = 0$.

b. Assume from now on that, in addition to the above, there exists a finite $C > 0$ so that

   $|\varepsilon_k| \le C,\ k = 1, 2, \ldots$. Prove

   (i) $S_\tau^2 \le (x + C)^2$.
   Hint. $S_\tau = S_{\tau-1} + \varepsilon_\tau$. What can be said about $S_{\tau-1}$?

   (ii) On the event $A^c = \{\max_{1 \le n \le m} |S_n| \le x\}$ we have $S_\tau^2 - \sigma_\tau^2 \le x^2 - \sigma_m^2$.

c. Prove
   (i) $0 = E\left(S_\tau^2 - \sigma_\tau^2\right) \le (x + C)^2 P(A) + (x^2 - \sigma_m^2) P(A^c)$

   (ii) $P(\max_{1 \le n \le m} |S_n| \le x) \le (x + C)^2 / \sigma_m^2$.

   Hint for (ii). Rearrange part (i) by replacing $P(A)$ by $1 - P(A^c)$, etc.