Preliminary Exam: Probability.

Time: 10:00am - 3:00pm, Friday, August 17, 2022

Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.

The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.
2. Start each problem on a new page.
3. Write only on one side of a page! There were cases in which the submission to D2L wasn't clear enough when the student used both sides of a page.

Problem 1. Let S be a random variable and let $x > 0$.

Prove the following:

a. (i). For any $t > 0$, $P(S > x) \leq e^{-tx} E(e^{tS})$.

(ii). $P(S > x) \leq \inf_{t>0} \{e^{-tx} E(e^{tS})\}$

b. Let $\varepsilon, \varepsilon_1, \dots, \varepsilon_n, n = 1,2, \dots$ be i.i.d. random variables with $P(\varepsilon = \pm 1) = \frac{1}{2}$. Let $S_n = \sum_{k=1}^{n} \varepsilon_k$.

(i) $E(e^{tS_n}) = (\frac{e^t + e^{-t}}{2})^n$

(ii) $P(S_n > x) \leq e^{-tx + nt^2/2}$

Hint. You can use without a proof the inequality $\frac{e^t + e^{-t}}{2} \leq e^{t^2/2}$

(It can be proved, for example, by Taylor expansion.)

c. $P(\frac{S_n}{\sqrt{n}} > x) \leq e^{-x^2/2}, n = 1,2, \dots$

Remark. Part c is known as Hoeffding's Inequality.

Problem 2. Let X, Y be random variables defined on the same probability space. We assume $E(X^4) + E(Y^2) < \infty$. Let $X_{a,b} \equiv aX + bX^2$, where $(a,b) \in R^2$.

a. Assume that $(a^*, b^*) \in R^2$ satisfy

$$\begin{cases} E\left((Y - X_{a^*,b^*})X\right) = 0 \\ E\left((Y - X_{a^*,b^*})X^2\right) = 0. \end{cases}$$

Calculate $E[(Y - X_{a^*,b^*})(X_{a^*,b^*} - X_{a,b})]$, where $(a,b) \in R^2$.

b. Prove by using part a. that $\min_{(a,b)\in R^2} E\left[\left(Y - X_{a,b}\right)^2\right] = E\left[\left(Y - X_{a^*,b^*}\right)^2\right]$.

Hint. $Y - X_{a,b} = (Y - X_{a^*,b^*}) + (X_{a^*,b^*} - X_{a,b})$

c. Let $\mathcal{H}$ be a family of random variables defined by $\mathcal{H} \equiv \{W \in \sigma\{X\}\colon\ E(W^2) < \infty\}$, where $\sigma\{X\}$ is the σ –algebra generated by X.

(i) Find $W^* \in \mathcal{H}$ so that

$$\min_{W\in\mathcal{H}} E[(Y - W)^2] = E[(Y - W^*)^2].$$

(ii) Determine which of the following two relationship is correct. Explain your reasoning!

(1) $E(Y - W^*)^2 \geq E\left(Y - X_{a^*,b^*}\right)^2$,

(2) $E(Y - W^*)^2 \leq E\left(Y - X_{a^*,b^*}\right)^2$.

Problem 3.

Let $\{X_k\}$ be independent random variables with $X_k \sim \text{Bernoulli}(\frac{1}{k}), k = 1, 2, \ldots$ and let $S_n = \sum_1^n X_k$. In what follows you may use the inequality:

$$\frac{1}{a} \geq \sum_{k=a}^{b} \frac{1}{k} - \ln(b/a) \geq 0,$$

where $a \leq b$ are positive integers.

a. $S_{2n} - S_n$ converge in distribution as $n \to \infty$.
Identify the limit distribution and prove the convergence.

b. $\frac{S_n - \ln(n)}{\sqrt{\ln(n)}}$ converge in distribution as $n \to \infty$.
Identify the limit distribution and prove the convergence.

c. $\frac{S_n}{\ln(n)}$ converge a.s. as $n \to \infty$. Identify the limit and prove the convergence.

Problem 4. Let $Z \sim N(0,1)$ and let $\{B(t):\ t \geq 0\}$ be a standard Brownian motion.

Prove the following.

a. (i) $P(Z > x) \leq e^{-\frac{x^2}{2}},\ x > 0.$

Hint: Look at Problem 1, part a.

(ii) $P(\max_{0 \leq s \leq t} \{B(s)\} > x) \leq 2 \cdot exp\{-\frac{x^2}{2 \cdot t}\},\quad x > 0,\ t > 0.$

For the rest of the problem, let $\alpha > 1$ and $t_n = \alpha^n,\ n = 1, 2, \ldots$

b. $\limsup_{n\to\infty} \frac{\max_{0 \leq s \leq t_n} \{B(s)\}}{\sqrt{2t_{n+1} \cdot loglog(t_{n+1})}} \leq 1, a.s.$

c. $\limsup_{n\to\infty} \max_{t_{n-1} \leq s \leq t_n} \{\frac{B(s)}{\sqrt{2s \cdot loglog(s)}}\} \leq \alpha \quad , a.s.$

Hint. Calculate $\lim_{n\to\infty} \frac{\sqrt{2t_{n+1} \cdot loglog(t_{n+1})}}{\sqrt{2t_{n-1} \cdot loglog(t_{n-1})}}$. What will be part b if it is done with $\frac{\max_{0 \leq s \leq t_n} \{B(s)\}}{\sqrt{2t_{n-1} \cdot loglog(t_{n-1})}}$?

Problem 5. Let $\{X_k, \mathcal{F}_k\}$ be a martingale sequence with $E(X_k^2) < \infty$, k= 0, 1, … Let

$$A_n^X = \sum_{k=1}^{n} E_{\mathcal{F}_{k-1}}(X_k - X_{k-1})^2, n = 1,2, \ldots$$

($\{A_n^X\}$ is known as "the **predictable** and increasing process associated with $\{X_k, \mathcal{F}_k\}$").

We assume that $A_1^X = C > 0$ where C is a constant.

Prove the following:

a. Let $Y_n = \sum_{k=1}^{n} \frac{X_k - X_{k-1}}{A_k^X}$, n=1,2, …

   (i) $E_{\mathcal{F}_{k-1}}(Y_k - Y_{k-1})^2 = \frac{A_k^X - A_{k-1}^X}{(A_k^X)^2} \le \frac{1}{C}$, a.s.

   Hint. $\{A_n^X\}$ is a predictable sequence of random variables.

   (ii) $\{Y_k, \mathcal{F}_k\}$ is a martingale with $E(Y_k^2) < \infty$, k= 1, 2, …

   For the rest of the problem define $A_n^Y = \sum_{k=1}^{n} E_{\mathcal{F}_{k-1}}(Y_k - Y_{k-1})^2$, $n = 2, 3, \ldots$ It follows from part a(i) that $A_n^Y = \sum_{k=1}^{n} \frac{A_k^X - A_{k-1}^X}{(A_k^X)^2}$.

b. (i) $A_n^Y \le \int_C^\infty x^{-2}dx = C^{-1}$, a.s.

   Hint. Draw the graph $f(x) = x^{-2}, x \ge C$ and see what the quantity $\sum_{k=1}^{n} \frac{A_k^X - A_{k-1}^X}{(A_k^X)^2}$ is on the graph.

   (ii) $\sup_{n=1,2,..} E(Y_n^2) < \infty$, and $\lim_{n\to\infty} Y_n$ exists and is finite a.s.

c. Show how to use Kronecker's Lemma to prove that if $A_n^X \xrightarrow[n\to\infty]{} \infty$, a.s. then $\frac{X_n}{A_n^X} \xrightarrow[n\to\infty]{} 0$, a.s.

Problem 6. Let $\{X, X_k\}$ be an i.i.d. sequence of random variables, k=1, 2, … and let

$S_n = \sum_1^n X_k, n = 1, 2, \dots, S_0 = 0$. Denote $d = E(e^X)$ and assume that $1 < d < \infty$.

Let $-\infty < a < 0 < b < \infty$, where a, b are constants. We define

$$T = \inf\{n \geq 1: S_n \leq a \text{ or } S_n \geq b\}.$$

a. Quote a theorem that leads to the result: $T < \infty, \text{a.s.}$ (Observe that X cannot be identically 0)

b. Let $\mathcal{F}_n = \sigma\{X_k, k = 1, \dots, n\}, n = 1, 2, \dots$ be the natural filtration of the sequence $\{X_k\}$, and let

$$Y_n = d^{-n} e^{S_n}, n = 1, 2, \dots.$$

(i) Prove that $\{Y_n, \mathcal{F}_n\}, n = 1, 2, \dots.$ is a martingale, and that $\lim_{n\to\infty} Y_n$ exists and is finite a.s.

(ii) Prove that T is a stopping time with respect to the filtration $\{\mathcal{F}_n\}$, and that $\{Y_{n\wedge T}, \mathcal{F}_n\}, n = 1, 2, \dots.$ is a martingale.

c. Assume that X is bounded (namely, $|X| < C, \text{a.s}$, where $C < \infty$ is a constant.)

(i) Prove that $|S_{n\wedge T}| \leq C + \max\{|a|, b\}, n = 1, 2, \dots.$

(ii) What is $E(Y_T)$? Prove your answer.