Preliminary Exam: Probability

Time: 10:00am - 3:00pm, Friday, August 27, 2021.

Your goal should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful, and complete.

The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!

Please make sure to apply the following guidelines:
1. On each page you turn in, **write on the top left your assigned code number** and on the **top right** the page number/total number of pages (m/n).
Don't write your name on any page.

2. Start each problem on a new page.

Problem 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space where $\Omega = [-1,1] \subset \mathbb{R}$, $\mathcal{F}$ represents the Borel sets contained in Ω and $P = \frac{\lambda}{2}$, where λ is the Lebesgue measure. In what follows W, X, Y are random variables which are defined on $(\Omega, \mathcal{F}, P)$. Let

$$\mathcal{G} = \{B \in \mathcal{F} : B = A \cup (A+1), A \subset [-1,0]\} \text{ , where } A+1 = \{\omega + 1 : \omega \in A\}.$$

Example: $A = (-\frac{2}{3}, -\frac{1}{3})$, $A+1 = \left(-\frac{2}{3}+1, -\frac{1}{3}+1\right) = \left(\frac{1}{3}, \frac{2}{3}\right)$, and $B = \left(-\frac{2}{3}, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, \frac{2}{3}\right)$.

You can assume without proof that $\mathcal{G} \subset \mathcal{F}$ is a σ algebra.

a. Let X be a random variable that satisfies $X(\omega) = X(\omega+1),\ \omega \in [-1,0]$. (Example $X(\omega) = \sin(2\pi\omega)$). Prove that $\{X \le x\} \in \mathcal{G}, x \in \mathbb{R}$, namely X is $\mathcal{G}$ measurable.

b. Let Y be a random variable. Prove that $\tilde{Y}(\omega) \equiv \begin{cases} \frac{Y(w)+Y(w+1)}{2}, & -1 \le \omega \le 0 \\ \frac{Y(w)+Y(w-1)}{2}, & 0 < \omega \le 1 \end{cases}$

is $\mathcal{G}$ measurable.

c. Let $W(\omega) = \begin{cases} 1, & -1 \le \omega \le -\frac{1}{2} \\ 0, & -\frac{1}{2} < \omega \le 1 \end{cases}$

What is $E(W|\,\mathcal{G})$? Verify that your answer is correct.

Problem 2. In this problem $\{B_t,\ t \geq 0\}$ represents a **Standard** Brownian Motion (SBM) defined on its canonical space $(\Omega = \mathrm{C}[0,\infty), \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, P)$.

Let $B_t^* = \max_{0\leq s\leq t}\{B_s\}, t \geq 0$ ("trailing maximum" of SBM). Observe that with this notation the reflection principle states: $B_t^* = |B_t|$ in distribution, for $t \geq 0$. Let $T > 0$ be fixed.

a. Let $Y_t = B_{T-t} - B_T,\ 0 \leq t \leq T$. Calculate $\mathrm{COV}(Y_t,\ Y_s), 0 \leq t, s \leq T$

b. Prove that $\{Y_t,\ 0 \leq t \leq T\} = \{B_t,\ 0 \leq t \leq T\}$ in distribution.

c. (i) Prove that $Y_T^* = B_T^* - B_T$, a. s.

(ii) Prove that $|B_T| = B_T^* - B_T$ in distribution.

Problem 3. In what follows $\mathrm{n} = 1, 2, \ldots$

Let $\{B_n\}$ be a sequence of events and $\{\mathcal{F}_n\}$ be a sequence of increasing σ algebras and assume that $B_n \in \mathcal{F}_n$.

Let $Y_n = \sum_{k=1}^{n} 1_{B_k}$ and $X_n = Y_n - \sum_{k=1}^{n} p_k(B_k)$, where $p_k(B_k) = E\left(1_{B_k} \middle| \mathcal{F}_{k-1}\right)$.

a. (i) Prove that $\{X_n , \mathcal{F}_n\}$ is a martingale.

(ii) Prove that $\{Y_n , \mathcal{F}_n\}$ is a submartingale and show its Doob's decomposition.

b. Prove that $\{X_n^2 - \sum_{k=1}^{n}(p_k(B_k) - p_k^2(B_k)) , \mathcal{F}_n\}$ is a martingale.

Hint: Observe that $\sum_{k=1}^{n}(p_k(B_k) - p_k^2(B_k))$ is $\mathcal{F}_{n-1}$ measurable.

c. (i). Prove that $\{X_n^2 , \mathcal{F}_n\}$ is a submartingale.

(ii). Show the Doob's decomposition of $\{X_n^2 , \mathcal{F}_n\}$.

Problem 4. Let $\{X, X_n, n = 1, 2, \dots\}$ be independent and identically distributed random variables so that X is symmetric, (i.e. $X = -X$ in distribution) and

$$P(|X| > x) = \begin{cases} 1 & if \quad 0 \le x \le 1 \\ x^{-\alpha} & if \quad x > 1 \end{cases}$$

where $0 < \alpha < 2$ is a constant.

Let $\{b_n \ge 0, n = 1, 2, \dots\}$ be a sequence of real numbers that satisfy $\sum_{n=1}^{\infty} b_n^{\alpha} < \infty$. Prove:

a. (i) $\sum_{n=1}^{\infty} P\left(|X| > \frac{1}{b_n}\right) < \infty$

Hint: what can you say about $\lim_{n\to\infty} \{\frac{1}{b_n}\}$?

(ii) $E\left(b_n X \cdot 1_{\{|X| \le \frac{1}{b_n}\}}\right) = 0$

b. $\sum_{n=1}^{\infty} E\left(b_n^2 X^2 \cdot 1_{\{|X| \le \frac{1}{b_n}\}}\right) < \infty$

Hint: Recall that $E(Y^2) = \int_0^{\infty} 2yP(|Y| \ge y)dy$ for any random variable Y.

c. $\sum_{n=1}^{\infty} b_n X_n$ converges a.s.

Problem 5. In what follows $\mathrm{n} = 1, 2, \ldots$

Let $\{ X_n \}$ be a sequence of uniformly bounded random variables, namely there exist a constant $C < \infty$ so that $| X_n | \leq C, a.s.$ In what follows $S_n = \sum_{k=1}^{n} \frac{X_k}{2^k}$.

a. (i) Prove that $S_n \underset{n\to\infty}{\longrightarrow} S$, a.s. where S is a random variable.

Hint: Why does $\left|\sum_{k=n}^{\infty} \frac{X_k}{2^k}\right| \underset{n\to\infty}{\longrightarrow} 0$ a.s. ?

(ii) Prove that $\varphi_{S_n}(t) \underset{n\to\infty}{\longrightarrow} \varphi_S(t), t \in \mathbb{R}$, where $\varphi_Y(t)$ denote the characteristic function of a random variable Y.

b. For the rest of the problem $\{X,\ X_n\}$ are independent and identically distributed with $P(X = -1) = P(X = 1) = 1/2$.

(i) Calculate $\varphi_{X_k/2^k}(t)$ and present it as a real-valued function. Do the same for $\varphi_{S_n}(t)$.

(ii) Present $\varphi_S(t)$ as an infinite product of real valued functions.

c. We continue with the setup of part b.

(i) Find $\varphi_Y(t)$ where $Y \sim \text{Uniform } (-1, 1)$.

(ii) Find the distribution of S.

Hint: You may use the identity: $\frac{\sin(t)}{t} = \prod_{k=1}^{\infty} \cos\left(\frac{t}{2^k}\right), t \in \mathbb{R}$ without proof.

Problem 6. Let $\{X, X_n, n = 1, 2, \ldots\}$ be independent and identically distributed random variables so that X is symmetric, (i.e. $X = -X$ in distribution) and

$$P(|X| > x) = \begin{cases} 1 \;\; if \quad 0 \le x \le e \\ \dfrac{e^2}{x^2 \ln(x)} \quad if \quad x > e \end{cases}$$

We will use the notations:

(1) $S_n = \sum_{k=1}^{n} X_k$, (2) $Y_{n,k} = X_k \cdot 1_{\{|X_k| \le \sqrt{n}\}}$, $k = 1, \ldots, n$, (3) $\tilde{S}_n = \sum_{k=1}^{n} Y_{k,n}$.

Prove:

a. (i) $P(S_n \neq \tilde{S}_n) \le nP(|X| > \sqrt{n}) \xrightarrow[n\to\infty]{} 0.$

(ii) $E(\tilde{S}_n) = 0,\; n = 1,2, \ldots$

b. (i) $E(X^2) = \infty$

Hints: $\int \frac{dx}{x ln(x)} = \ln(\ln(x)) + C.$

See also the hint in problem 4, part b.

(ii) $\frac{E(Y_{n,1}{}^2)}{2e^2 \ln(\ln(n))} \xrightarrow[n\to\infty]{} 1$

Hint: $P(|Y_{n,1}| > x) = P(|X| > x) - P(|X| > \sqrt{n}),\;\; 0 < x \le \sqrt{n}$

c. (i) $\frac{\tilde{S}_n}{a_n}$ converges in distribution to N(0, 1) as $n \to \infty$, where

$a_n = \sqrt{2e^2 \text{n} \ln(\ln(n))}.$

(ii) $\frac{S_n}{a_n}$ converges in distribution to N(0, 1) as $n \to \infty$.