Master's Exam – Spring 2023
March 28, 2023
8:00 a.m. – 12:00 p.m.

PID Number: ______________________

A. The number of points for each problem is given.
B. There are 7 problems with varying number of parts.

   Problems 1 – 4   Probability (60 points)

   Problems 5 – 7   Statistics (55 points)

C. Write your answers on the exam paper itself. If you need more room you may use the extra sheets provided. Answer as many questions as you can. You may use calculator and formula sheet.

1. (15 points) A set of 200 people consisting of 100 men and 100 women is randomly divided into 1000 pairs of 2 each. Given an upper bound to the probability that at most 30 of these pairs will consist of a man and a woman using chebyshev's inequality.

2. Let (X, Y) denote a uniformly chosen random point inside the unit square with vertices (0, 0), (1, 0), (1, 1) and (0, 1), that is,

$$[0,1]^2 = [0,1] \times [0,1] = \{(x, y): 0 \leq x, y \leq 1\}.$$

Let $0 \leq a < b \leq 1$.

a) (5 points) Find $P(a < X < b)$, that is, the probability that the x −coordinate X of the chosen point lies in the interval (a, b).

b) (10 points) Compute $P(|X - Y| \leq \frac{1}{4})$

3. The random vector *(U, V)* has the probability density function

$$g(u,v) = \begin{cases} 2e^{-(u+v)}, & 0 \le u \le v < \infty \\ 0, & otherwise \end{cases}$$

(a) (7 pts.) Find the probability density of *U.*

(b) (8 pts.) Find the probability density of $U + V$.

4. Let Z_n be the discrete random variable which puts a mass pf $\frac{n+1}{5n}$ on $\frac{1}{n}$ and a mass of $\frac{4n-1}{5n}$ on $1-\frac{1}{n}$.

   (a) (5 pts) Calculate the moment generating function of Z_n

   (b) (10 pts) Using (a) or otherwise, show that Z_n converges in distribution to a Bernoulli random variable with parameter 0.8

5. Let $X_1, X_2, \ldots, X_n$ be iid with probability $h_\theta(x) = \theta x^{\theta-1} I[x \in (0,1)], \theta > 0$.
   (a) (9 points) Find the maximum likelihood estimator (mle) of θ.
   (b) (6 points) Find the asymptotic distribution of the maximum likelihood estimator.

6. Let $(x_1, x_2, \dots, x_{25})$ be the values observed when a random sample of size 25 is taken from a normal population with mean μ, and variance, $\sigma^2 = 100$, known.
   (a) (5 points) Give the uniformly most powerful $\alpha = 0.05$ test for $H_o: \mu = 40$ versus $H_1: \mu > 40$. Just give the test, you need not prove that it is UMP.
   (b) (5 points) Give the power of the test for $\mu = 45$
   (c) (5 points) What should the sample size n be so that the power at $\mu = 42$ is 0.80?
   (d) (5 points) Suppose that σ was not known, and that $\bar{X}$ was observed to be 47.4 and the sample variance, S^2, was 81.0. Find a 90% confidence interval for μ

7. Consider the linear regression model $Y_i = \beta x_i + e_i,\ i = 1, 2, \dots, n$ where the x_i are constants and the random error e_i are independent with $E(e_i) = 0, Var(e_i) = \sigma^2$.
   (a) (8 points) Derive the least square estimate for the slope β.
   (b) (12 points)Derive the mean and variance for the estimate in (a) above.