Master's Exam – SPRING 2019
March 19, 2019
1:00 p.m. – 5:00 p.m.

PID Number: ________________________

A. The number of points for each problem is given.
B. There are 8 problems with varying number of parts.
   Problems 1 – 4 Probability (60 points)
   Problems 5 – 8 Statistics (60 points)

C. Write your answers on the exam paper itself. If you need more room you may use the extra sheets provided. Answer as many questions as you can. Tables are provided.

1. (10 points) Suppose that we have a medical test that detects a particular disease 96% of the time, but gives false positives 2% of the time. Assume that 0.5% of the population carries the disease. If a random person test positive for the disease, what is the probability that they actually **do not** carry the disease?

2. Let (X, Y) denote a uniformly chosen random point inside the unit square with vertices (0, 0), (1, 0), (1, 1) and (0, 1), that is,

$$[0,1]^2 = [0,1] \times [0,1] = \{(x,y): 0 \leq x, y \leq 1\}.$$

Let $0 \leq a < b \leq 1$.

a) (5 points) Find $P(a < X < b)$, that is, the probability that the x −coordinate X of the chosen point lies in the interval (a, b).

b) (10 points) Compute $P(|X - Y| \leq \frac{1}{4})$

3. The random vector *(U, V)* has the probability density function

$$g(u,v) = \begin{cases} 2e^{-(u+v)}, & 0 \le u \le v < \infty \\ 0, & otherwise \end{cases}$$

(a) (7 pts.) Find the probability density of *U.*

(b) (8 pts.) Find the probability density of $U + V$.

4. Let $\{X_n\}, n = 1,2, \dots$ be a sequence of independent exponential distributed random variables where X_n has mean $n, n = 1,2, \dots$ Thus, the cumulative distribution function of X_n is $F_n(x) = 1 - e^{-\frac{x}{n}}, x > 0$. Define $Z_n \coloneqq \min\{X_1, X_2, \dots, X_n\}, n = 1,2, \dots$

   (a) (8 points) Determine the probability density function for Z_n
   (b) (7 points) Prove that $Z_n \to 0$ in probability as $n \to \infty$.
   (c) (5 points) Suppose $T_n \coloneqq \min\{X_1, X_2, \dots, X_n\}$. Find the probability distribution for T_n
   (Note: You **do not** have to simplify the distribution in part c.)

5. Let $X_1, X_2, \dots, X_n$ be iid with probability $h_\theta(x) = \theta x^{\theta-1} I[x \in (0,1)], \theta > 0$.
   (a) (9 points) Find the maximum likelihood estimator (mle) of θ.
   (b) (6 points) Find the asymptotic distribution of the maximum likelihood estimator.

6. Let $(x_1, x_2, \dots, x_{25})$ be the values observed when a random sample of size 25 is taken from a normal population with mean μ, and variance, $\sigma^2 = 100$, known.
   (a) (5 points) Give the uniformly most powerful $\alpha = 0.05$ test for $H_o: \mu = 40$ versus $H_1: \mu > 40$. Just give the test, you need not prove that it is UMP.
   (b) (5 points) Give the power of the test for $\mu = 45$
   (c) (5 points) What should the sample size n be so that the power at $\mu = 42$ is 0.80?
   (d) (5 points) Suppose that σ was not known, and that $\bar{X}$ was observed to be 47.4 and the sample variance, S^2, was 81.0. Find a 90% confidence interval for μ

7. Consider the linear regression model $Y_i = \beta x_i + e_i,\ i = 1, 2, \dots, n$ where the x_i are constants and the random error e_i are independent with $E(e_i) = 0, Var(e_i) = \sigma^2$.
   (a) (8 points) Derive the least square estimate for the slope β.
   (b) (12 points)Derive the mean and variance for the estimate in (a) above.

8. (5 points) Consider a fitted regression line of $Y on\ X$ with model : $Y = \beta_o + \beta_1 X + \epsilon_i$
State whether the following statements are True or False (Circle your choices)

   a. The sum of the residuals is zero. True or False

   b. The regression line always passes through $(\bar{x}, \bar{y})$. True or False

   c. The regression line always passes through the point (0, 0). True or False

   d. The intercept of the regression line always lies between -1 and I. True or False

   e. The sum of the observed values Y_i is equal to the sum of the fitted values $\widehat{Y_i}$ True or False