Master’s Exam – Fall 2018
October 11, 2018
10:00 a.m. – 12:00 p.m.

Number: ________________________

A. The number of points for each problem is given.
B. There are 5 problems with varying number of parts.
   1 – 5 Probability (65 points)

C. Write your answers on the exam paper itself. If you need more room you may use the extra sheets provided. Answer as many questions as you can. Tables are provided.

1. (10 points) Suppose that we have a medical test that detects a particular disease 96% of the time, but gives false positives 2% of the time. Assume that 0.5% of the population carries the disease. If a random person test positive for the disease, what is the probability that they actually carry the disease?

2. Let (X, Y) denote a uniformly chosen random point inside the unit square with vertices (0, 0), (1, 0), (1, 1) and (0, 1), that is,

$$[0,1]^2 = [0,1] \times [0,1] = \{(x,y): 0 \leq x, y \leq 1\}.$$

Let $0 \leq a < b \leq 1$.

   a) (5 points) Find $P(a < X < b)$, that is, the probability that the x −coordinate X of the chosen point lies in the interval (a, b).
   b) (10 points) Compute $P(|X - Y| \leq \frac{1}{4})$

3. Let Z_n be the discrete random variable which puts a mass of $\frac{n+1}{5n}$ on $\frac{1}{n}$ and a mass of $\frac{4n-1}{5n}$ on $1-\frac{1}{n}$.

(a) (5 pts.) Calculate the moment generating function of Zn.

(b) (10 pts.) Using (a) or otherwise, show that Zn converges in distribution to a Bernoulli random variable with parameter 0.8.

4. (10 points) The geometric distribution is said to have the memoryless property which means that there are no successes in the first n trials, the probability that the first success comes at trial $n + k$ is the same as a freshly started sequence of trials yield the first success at the Kth trial.

   Show that if X has geometric distribution with parameter p then

$$P(X = n + k | X > n) = P(X = k) \quad for\, n, k \geq 1$$

5. Let $\{X_n\}, n = 1,2, \ldots$ be a sequence of independent exponential distributed random variables where X_n has mean $n, n = 1,2, \ldots$ Thus, the cumulative distribution function of X_n is $F_n(x) = 1 - e^{-\frac{x}{n}}, x > 0$. Define $Z_n \coloneqq \min\{X_1, X_2, \ldots, X_n\}, n = 1,2, \ldots$

(a) (8 points) Determine the probability density function for Z_n

(b) (7 points) Prove that $Z_n \to 0$ in probability as $n \to \infty$.

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