Extremes of Autoregressive Threshold Processes

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joint work with

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Overview

- Introduction: Model and Assumptions

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- Regularly Varying Noise
  - TAR(S,q) Model: O-regular Variation
  - TAR(S,1) Model: Regular Variation
  - TAR(S,1) Model: Extremal Behavior

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  - TAR(S,q) Model: O-regular Variation
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- Semi-Heavy Tailed Noise
  - TAR(S,q) Model: Tail Behavior
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TAR (**T**hreshold **A**uto**R**egressive) Model

- $(Z_k)_{k \in \mathbb{N}_0}$ be an iid sequence
- $\{J_i : i = 1, \ldots, S\}$ be a partition of $\mathbb{R}^p$
- α_i and β_{ij} real constants

A TAR(S,q) process has the representation

$$X_k = \begin{cases} \alpha_1 + \beta_{11} X_{k-1} + \ldots + \beta_{1q} X_{k-q} + Z_k, & (X_{k-1}, \ldots, X_{k-p}) \in J_1, \\ \alpha_2 + \beta_{21} X_{k-1} + \ldots + \beta_{2q} X_{k-q} + Z_k, & (X_{k-1}, \ldots, X_{k-p}) \in J_2, \\ \vdots & \\ \alpha_S + \beta_{S1} X_{k-1} + \ldots + \beta_{Sq} X_{k-q} + Z_k, & (X_{k-1}, \ldots, X_{k-p}) \in J_S. \end{cases}$$

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$\Longrightarrow$ combination of **AR(q)** processes governed by S different **regimes**

TAR Model

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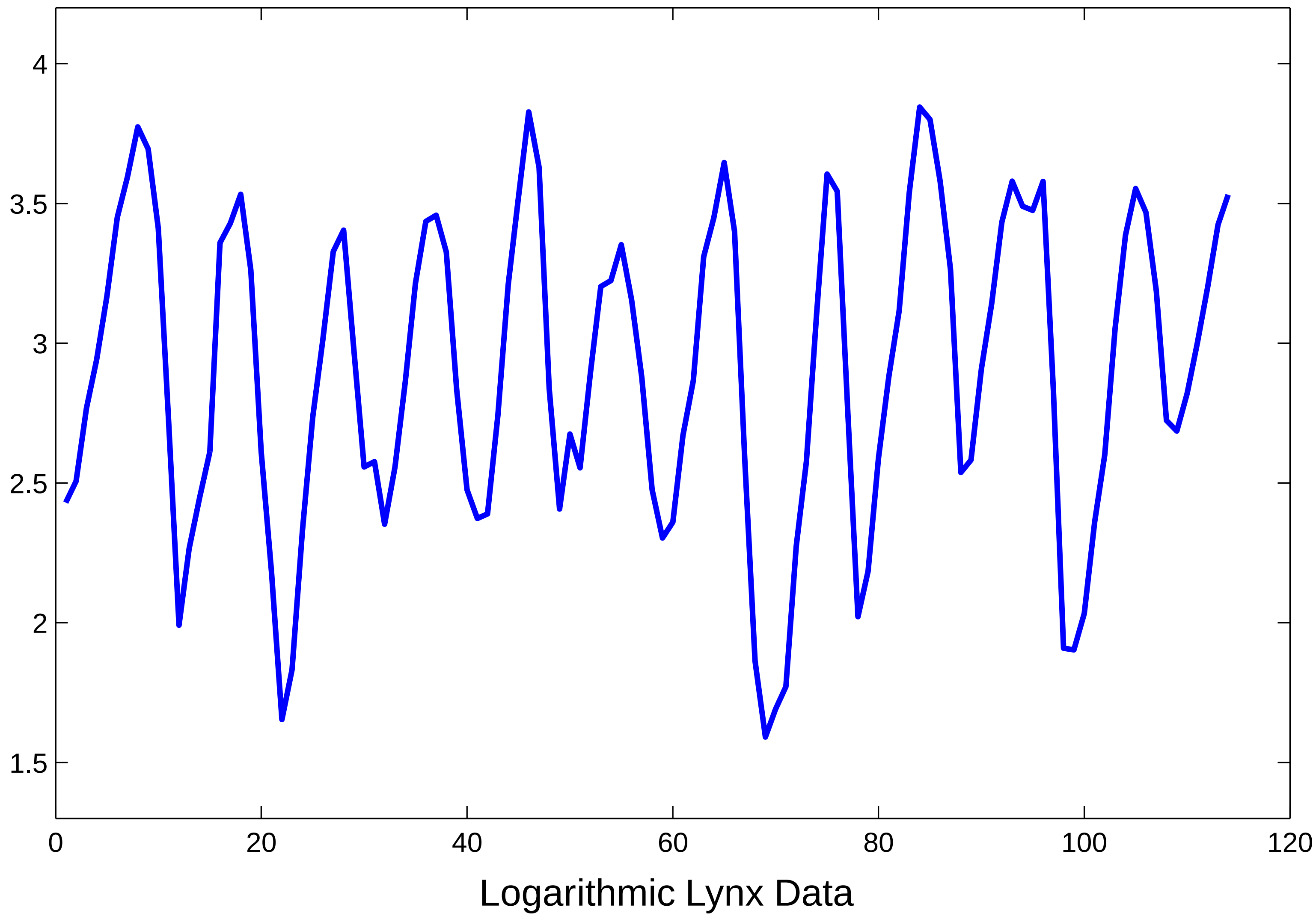
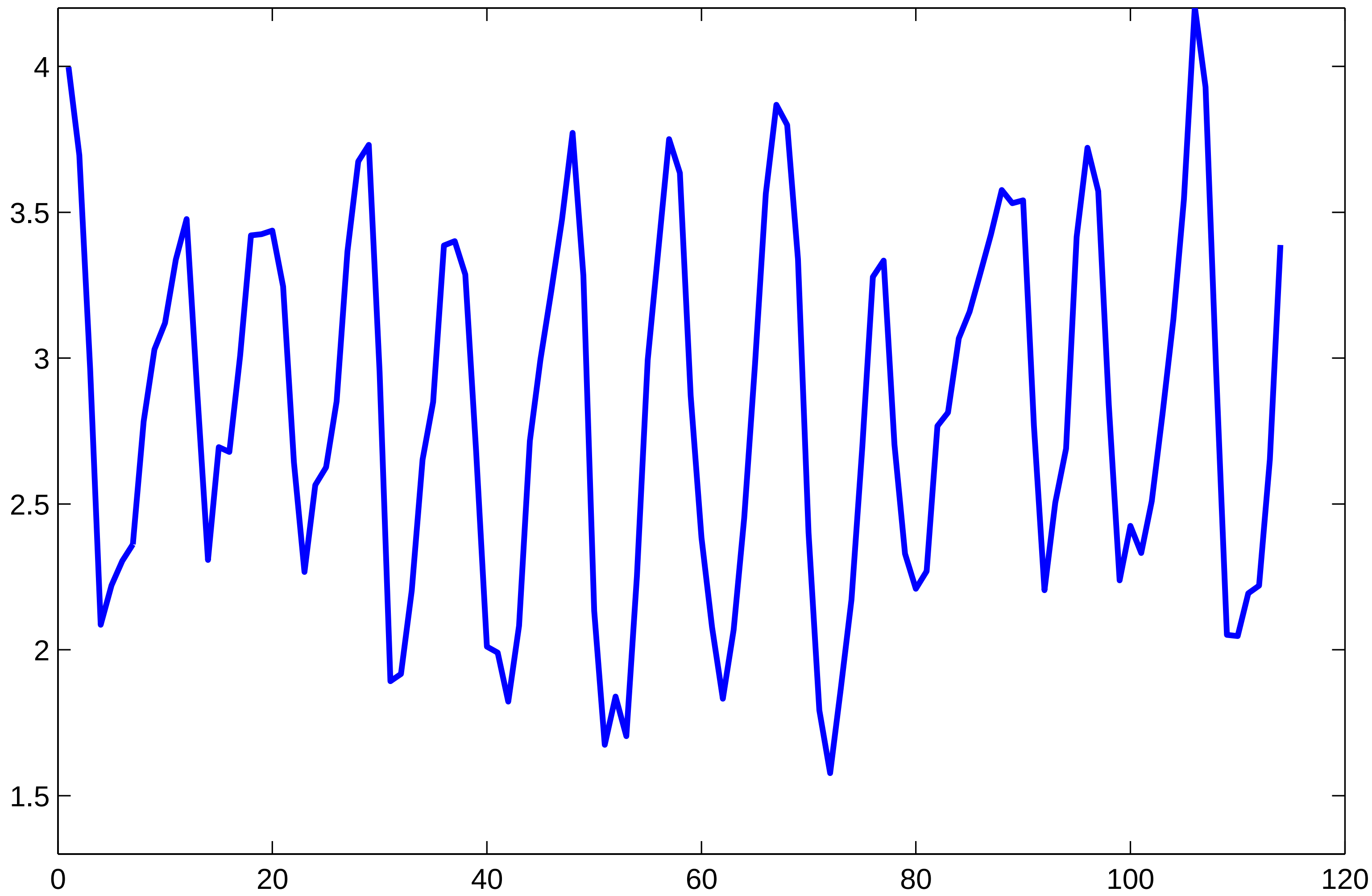
Remark:

TAR models proposed by Tong (1977) and fitted to lynx data. Can for example explain seasonality.

Canadian Lynx Data from 1821-1934



TAR(2,7) model



$$X_k = \begin{cases} 0.546 + 1.032X_{k-1} - 0.173X_{k-2} + 0.171X_{k-3} - 0.431X_{k-4} + \ldots & \\ \ldots + 0.332X_{k-5} - 0.284X_{k-6} + 0.21X_{k-7} + Z_k^{(1)}, & \text{if } X_{k-2} \leq 3.116 \\ 2.632 + 1.492X_{k-1} - 1.324X_{k-2} + Z_k^{(2)}, & \text{if } X_{k-2} > 3.116 \end{cases}$$

Assumptions on the TAR Model

Noise distribution condition (DC)

- The marginal distribution of Z_1 has a Lebesgue density h satisfying $\inf_{x\in K} h(x) > 0$ for every compact set $K \subset \mathbb{R}$.

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$\alpha := \max_{k=1,\ldots,S} |\alpha_k|$ and $\beta_j := \max_{i=1,\ldots,S} |\beta_{ij}|$

Conclusion

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- $(X_k)_{k\in\mathbb{N}_0}$ has a unique strict stationary solution
- $(X_k)_{k\in\mathbb{N}_0}$ is strongly mixing with geometrically decreasing mixing rate

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Key Lemma

Let $(\widetilde{Z}_k)_{k\in\mathbb{Z}}$ be i. i. d. with $\widetilde{Z}_k = |Z_k| + \alpha$ and

$$\boldsymbol{\widetilde{X}_k := \sum_{j=1}^{q} \beta_j \widetilde{X}_{k-j} + \widetilde{Z}_k}$$

be stationary.

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be stationary. Then

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  where for some $0 < \gamma < 1$ and $K > 0$

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- $\widetilde{X}_k$ has a heavier tail than $|X_k|$:

$$\mathbb{P}(|\boldsymbol{X}_{k_1}| > \boldsymbol{x}_1, \ldots, |\boldsymbol{X}_{k_m}| > \boldsymbol{x}_m) \leq \mathbb{P}(\widetilde{\boldsymbol{X}}_{k_1} > \boldsymbol{x}_1, \ldots, \widetilde{\boldsymbol{X}}_{k_m} > \boldsymbol{x}_m)$$

Comparison

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If $(X_0^*, \ldots, X_{q-1}^*) = (|X_0|, \ldots, |X_{q-1}|)$ and

$$X_k^* = \alpha + \beta_1 X_{k-1}^* + \ldots + \beta_q X_{k-q}^* + Z_k,$$

then $$|X_k| \leq X_k^*.$$

Assumptions on the TAR Model

Tail balance condition (TB)

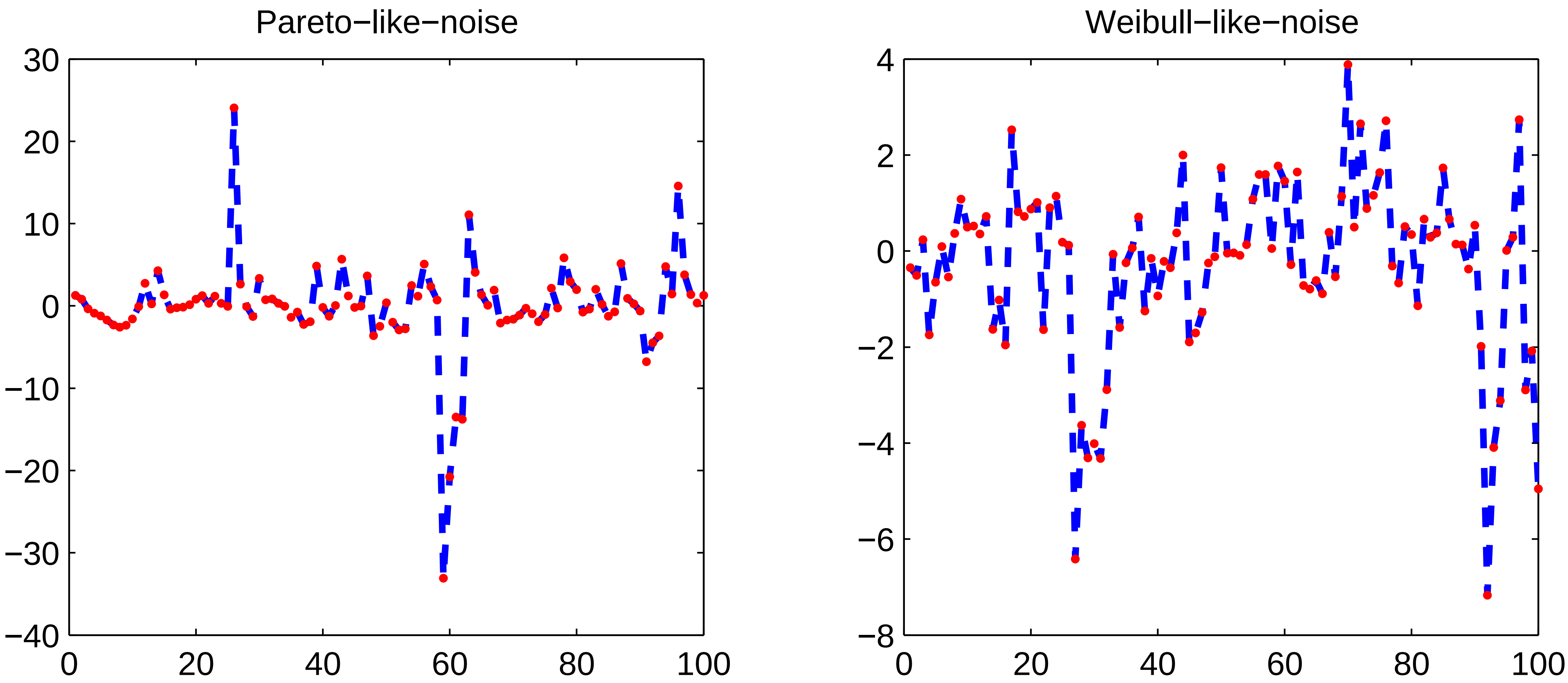
Let $p^+, p^- \in [0, 1]$ such that

$$p^+ + p^- = 1$$

and

$$\begin{aligned} \mathbb{P}(Z_1 > x) &\sim p^+ \mathbb{P}(|Z_1| > x) \text{ as } x \to \infty, \\ \mathbb{P}(Z_1 < -x) &\sim p^- \mathbb{P}(|Z_1| > x) \text{ as } x \to \infty. \end{aligned}$$

Example



$$X_k = \begin{cases} 0.7X_{k-1} + Z_k, & X_{k-1} \leq 0, \\ 0.2X_{k-1} + Z_k, & X_{k-1} > 0. \end{cases}$$

Regular Varying Noise

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f regularly var. $\Longrightarrow$ f O-regularly var.

Assumption

Assumption: $|Z_1| \in \mathcal{R}_{-\kappa}$ for some $\kappa > 0$, i. e.

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- stable distribution
- Pareto distribution
- Cauchy distribution

Lemma

Suppose

- $(X_k)_{k\in\mathbb{N}_0}$ be a stationary **TAR(S,q)** process.
- TB and DC hold.
- $|\boldsymbol{Z_1}| \in \boldsymbol{\mathcal{R}_{-\kappa}}$.

Lemma

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Then $\mathbb{P}(|\boldsymbol{X_0}| > \boldsymbol{x})$ as $x \to \infty$ is O-regularly varying,

$$\begin{aligned} \mathbf{2}^{-\kappa} &\leq \liminf_{x\to\infty} \frac{\mathbb{P}(|\boldsymbol{X_0}| > \boldsymbol{x})}{\mathbb{P}(|\boldsymbol{Z_1}| > \boldsymbol{x})} \\ &\leq \limsup_{x\to\infty} \frac{\mathbb{P}(|\boldsymbol{X_0}| > \boldsymbol{x})}{\mathbb{P}(|\boldsymbol{Z_1}| > \boldsymbol{x})} \leq \sum_{j=0}^{\infty} \psi_j^{\kappa} \end{aligned}$$

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be stationary. Then

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$$
\begin{aligned}
2^{-\kappa} &\le \liminf_{x\to\infty} \frac{\mathbb{P}(|X_0| > x)}{\mathbb{P}(|Z_1| > x)} \\
&\le \limsup_{x\to\infty} \frac{\mathbb{P}(|X_0| > x)}{\mathbb{P}(|Z_1| > x)} \le \sum_{j=0}^{\infty} \psi_j^{\kappa} < \infty
\end{aligned}
$$

Counter-example

We assume

- $\mathbb{P}(Z_1 > x) \sim x^{-\kappa}$ as $x \to \infty$ for some $\kappa > 0$.
- TB and DC hold.
- $J_1 := \bigcup_{m \in \mathbb{N}_0} \left(4^m, 4^{m+1/2}\right]$, $J_2 := \mathbb{R} \setminus J_1$.
- $0 < \beta < 1$.

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The tail of the stationary solution of the TAR(2,1) model

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is O-regularly varying

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is O-regularly varying but not regularly varying.

TAR(S,1)-Model

We investigate the TAR(S,1)-model

$$\boldsymbol{X_k = \sum_{i=1}^{S} \{\alpha_i + \beta_i X_{k-1}\} \, 1_{\{X_{k-1} \in J_i\}} + Z_k}$$

where

- For some $r_1, r_2 \in \mathbb{R}$, $r_1 \leq r_2$:

$$\boldsymbol{J_1 = (-\infty, r_1]} \quad \text{and} \quad \boldsymbol{J_2 = (r_2, \infty).}$$

- $\{J_i : i = 3, \ldots, S\}$ is a measurable partition of $(r_1, r_2]$

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The TAR(S,1) process has the representation

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Then $|X_0| \in \mathcal{R}_{-\kappa}$ and

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_0 > x)}{\mathbb{P}(|Z_1| > x)} = \frac{p^+ + p^-(\beta_1^-)^\kappa}{1-(\beta_2^+)^\kappa - (\beta_1^-)^\kappa(\beta_2^-)^\kappa} =: \widetilde{p}^+$$

and

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_0 < -x)}{\mathbb{P}(|Z_1| > x)} = \frac{p^- + p^+(\beta_2^-)^\kappa}{1-(\beta_1^+)^\kappa - (\beta_2^-)^\kappa(\beta_1^-)^\kappa} =: \widetilde{p}^-.$$

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and

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Multivariate Regular Variation

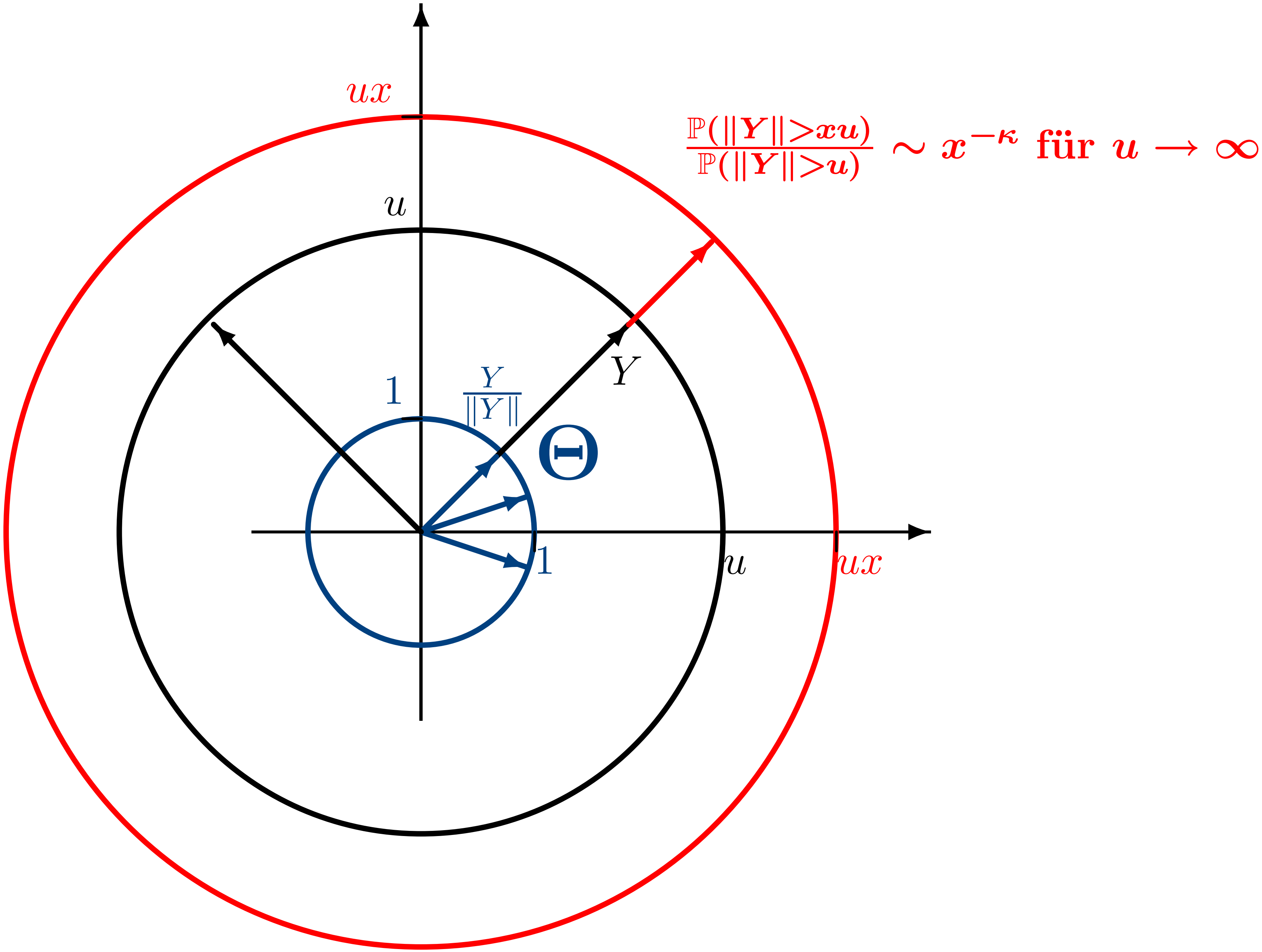
Definition

$\mathbf{Y} \in \mathbb{R}^m$ is multivariate regularly varying with index $-\kappa < 0$ if there exists a random vector Θ with values in $\mathbb{S}^{m-1}$ such that for every $x > 0$,

$$\frac{\mathbb{P}(\|\mathbf{Y}\| > ux, \mathbf{Y}/\|\mathbf{Y}\| \in \cdot)}{\mathbb{P}(\|\mathbf{Y}\| > u)} \xrightarrow{w} x^{-\kappa}\mathbb{P}(\boldsymbol{\Theta} \in \cdot)$$

as $u \to \infty$ on $\mathcal{B}(\mathbb{S}^{m-1})$

Multivariate Reguläre Variation



Notation

$$X_k = \alpha_1 + \beta_1 X_{k-1} + Z_k \text{ for } X_{k-1} \in (-\infty, r_1]$$

$$X_k = \alpha_2 + \beta_2 X_{k-1} + Z_k \text{ for } X_{k-1} \in (r_2, \infty)$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \beta_2^+ & \beta_1^- \\ \beta_2^- & \beta_1^+ \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

and for $m \in \mathbb{N}_0$ define

$$\widetilde{\mathbf{C}}^{(m)} = \begin{pmatrix} I & 0 & 0 & \cdots & 0 \\ B & I & 0 & \ddots & \vdots \\ B^2 & B & I & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ B^m & B^{m-1} & \cdots & B & I \end{pmatrix} \in \mathbb{R}^{2(m+1)\times 2(m+1)}$$

Notation

Further, for $m \in \mathbb{N}_0$ define

$$\mathbf{S}^{(m)} = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \in \mathbb{R}^{(m+1)\times 2(m+1)}$$

and

$$\begin{aligned} \mathbf{C}^{(m)} &:= \mathbf{S}^{(m)}\widetilde{\mathbf{C}}^{(m)} \\ &=: (\mathbf{c}_0^+, \mathbf{c}_0^-, \ldots, \mathbf{c}_m^+, \mathbf{c}_m^-) \in \mathbb{R}^{(m+1)\times 2(m+1)}. \end{aligned}$$

Multivariate Regular Variation

Theorem

Suppose

- $(X_k)_{k\in\mathbb{N}_0}$ be a stationary **TAR(S,1)** process.
- TB and DC hold.
- $|Z_1| \in \mathcal{R}_{-\kappa}$.

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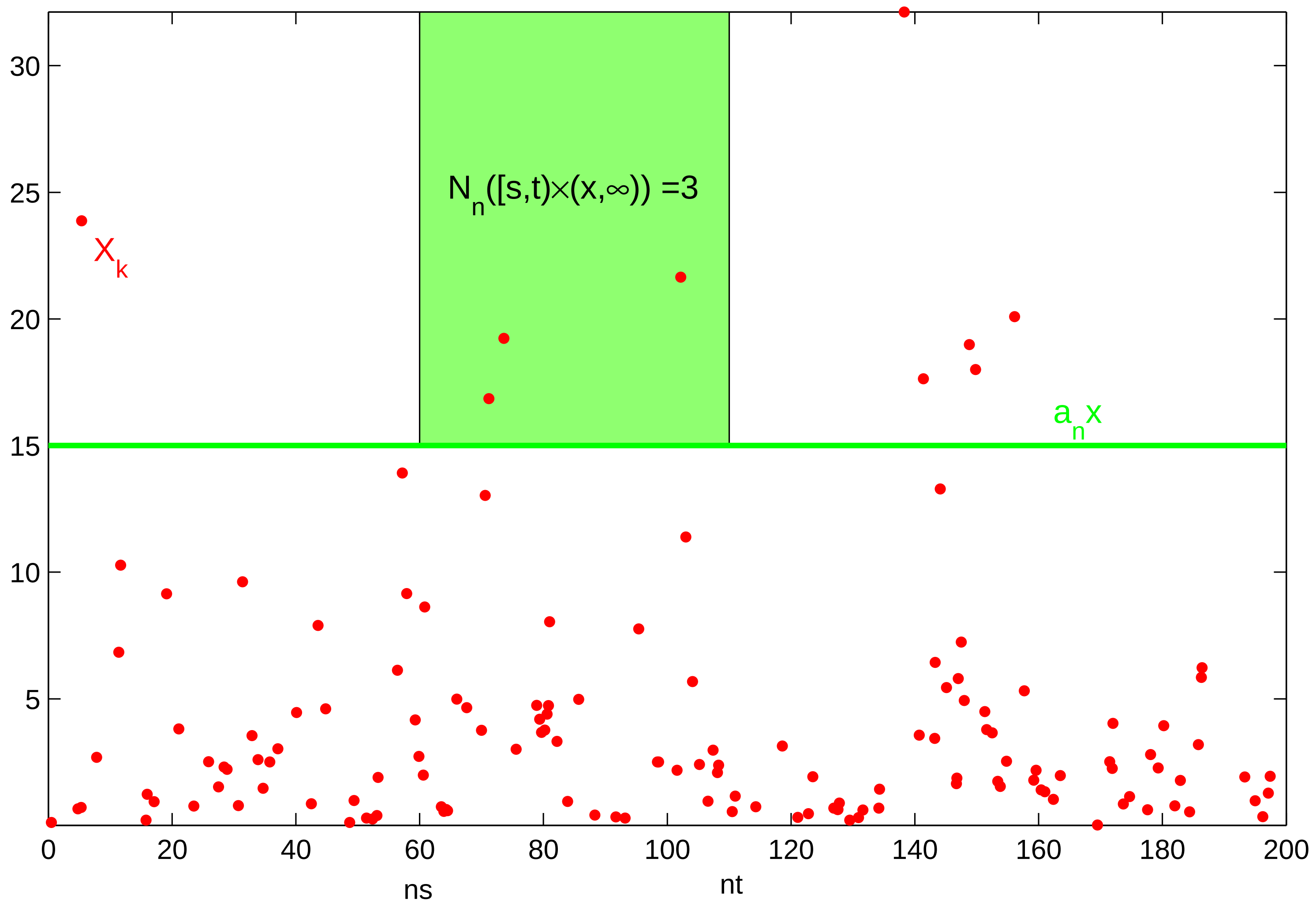
Then $\mathbf{X}^{(m)} = (X_0, \ldots, X_m)^T$ is multivariate regularly varying with index $-\kappa$, and spectral measure

$$\mathbb{P}(\Theta^{(m)} \in \cdot) = \frac{1}{\widetilde{p}^+ + \widetilde{p}^- + m}\left[\widetilde{p}^+ \mathbf{1}_{\{c_0^+ \in \cdot\}} + \widetilde{p}^- \mathbf{1}_{\{c_0^- \in \cdot\}}\right] + \frac{1}{\widetilde{p}^+ + \widetilde{p}^- + m}\left[\sum_{j=1}^{m}\left(p^+ \mathbf{1}_{\{c_j^+ \in \cdot\}} + p^- \mathbf{1}_{\{c_j^- \in \cdot\}}\right)\right]$$

Point process

$$N_n = \sum_{k=1}^{\infty} \varepsilon_{\left(\frac{k}{n}, a_n^{-1} X_k\right)}$$



Theorem: Point Process Convergence

- $(X_k)_{k\in\mathbb{N}_0}$ be a stationary **TAR(S,1)** process.
- TB and DC hold.
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- $0 < a_n \uparrow \infty$ be a sequence of constants such that

$$\lim_{n\to\infty} n\mathbb{P}(|X_0| > a_n) = 1.$$

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$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1} X_k)} \overset{w}{\Longrightarrow} \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \varepsilon_{(s_k, b_1^{(j)} P_k^+ + b_2^{(j)} P_k^-)}$$

Notation

Write

$$B^j = \begin{pmatrix} \beta_2^+ & \beta_1^- \\ \beta_2^- & \beta_1^+ \end{pmatrix}^j = \begin{pmatrix} b_{11}^{(j)} & b_{12}^{(j)} \\ b_{21}^{(j)} & b_{22}^{(j)} \end{pmatrix} \in \mathbb{R}^{2\times 2},\ j \in \mathbb{N}_0.$$

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Putting

$$b_1^{(j)} := b_{11}^{(j)} - b_{21}^{(j)} \quad \text{and} \quad b_2^{(j)} := b_{12}^{(j)} - b_{22}^{(j)}$$

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where $\sum_{k=1}^{\infty} \varepsilon_{(s_k, P_k)}$ is $\mathbf{PRM}(\vartheta)$ on $[0,\infty) \times (\overline{\mathbb{R}} \setminus \{0\})$ with

$$\vartheta(dt \times dx) = dt \times \kappa(\widetilde{p}^{+} + \widetilde{p}^{-})^{-1} \times \ldots$$

$$\ldots \times (p^{+} x^{-\kappa-1} \mathbf{1}_{(0,\infty)}(x) + p^{-}(-x)^{-\kappa-1} \mathbf{1}_{(-\infty,0)}(x))\, dx$$

Corollary: Running Maxima

Let further

- $p^+ > 0$
- $0 < \widetilde{a}_n \uparrow \infty$ be a sequence of constants such that

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Then

$$\lim_{n\to\infty} \mathbb{P}(\tilde{a}_n^{-1} \max(X_1, \ldots, X_n) \leq x)$$
$$= \exp\left(-(1 - (\beta_2^+)^\kappa - (\beta_1^-)^\kappa(\beta_2^-)^\kappa)x^{-\kappa}\right)$$

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Remark:

$$\theta = 1 \iff (\beta_2 \le 0 \text{ and } \beta_1 \ge 0) \text{ or } \beta_2 = 0$$

Then no extremal clusters occur.

Corollary: Cluster Sizes

$$\sum_{k=1}^{\infty} \varepsilon_{(k/n, a_n^{-1} X_k)}(\cdot \times (x, \infty)) \stackrel{w}{\Longrightarrow} \sum_{k=1}^{\infty} \zeta_k \varepsilon_{\widetilde{s}_k} \text{ as } n \to \infty$$

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- $(\widetilde{s}_k)_{k \in \mathbb{N}}$ be the jump times of a Poisson process with intensity $(\widetilde{p}^+ + \widetilde{p}^-)^{-1}(p^+ + p^-(\beta_1^-)^\kappa)x^{-\kappa}$.

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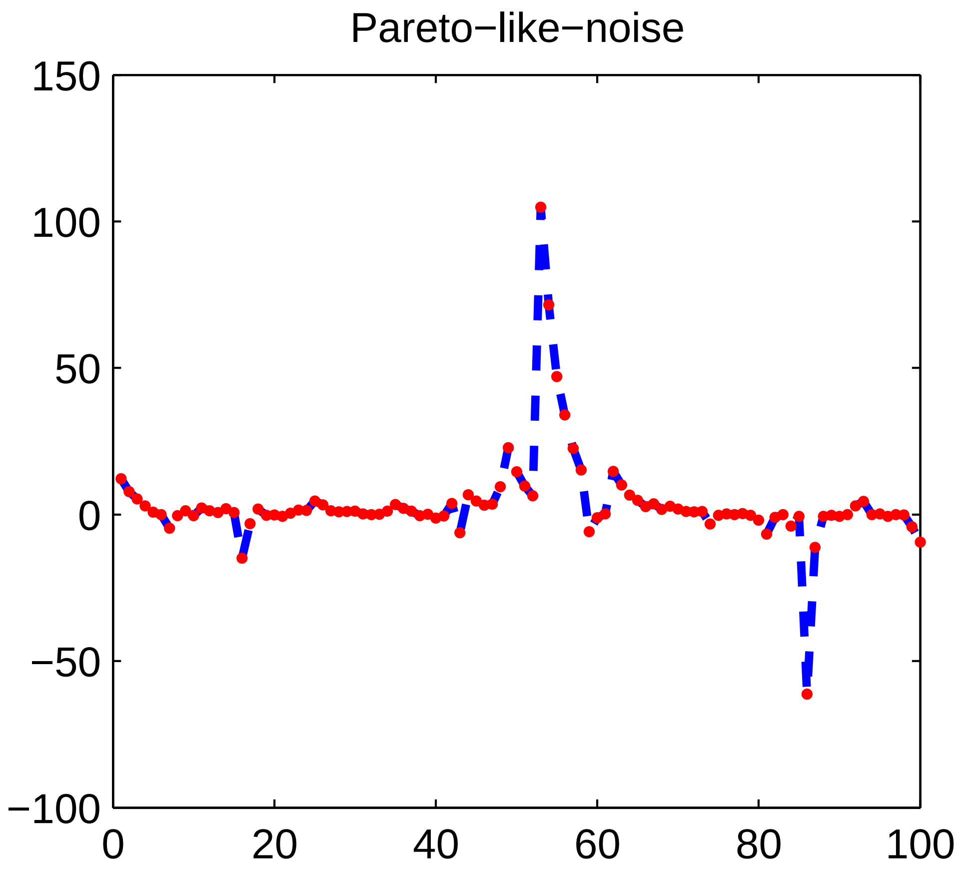
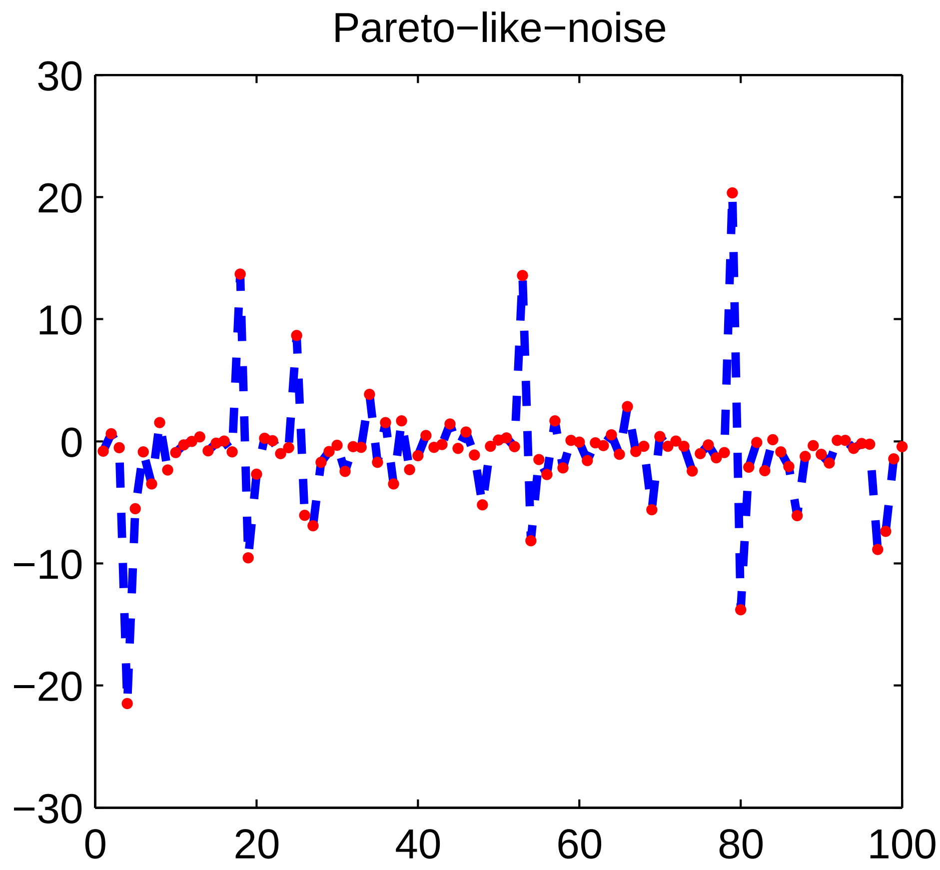
- $(\widetilde{s}_k)_{k \in \mathbb{N}}$ be the jump times of a Poisson process with intensity $(\widetilde{p}^+ + \widetilde{p}^-)^{-1}(p^+ + p^-(\beta_1^-)^\kappa) x^{-\kappa}$.
- $(\zeta_k)_{k \in \mathbb{N}}$ be an iid sequence independent of $(\widetilde{s}_k)$ with

$$\mathbb{P}(\zeta_1 = j) = \frac{p^+((\widetilde{b}_1^{(j-1)})^\kappa - (\widetilde{b}_1^{(j)})^\kappa) + p^-((\widetilde{b}_2^{(j-1)})^\kappa - (\widetilde{b}_2^{(j)})^\kappa)}{p^+ + p^-(\beta_1^-)^\kappa}$$

where $1 = \widetilde{b}_1^{(0)} \geq \widetilde{b}_1^{(1)} \geq \ldots$ is the order statistic of $(\max\{0, b_1^{(j)}\})_{j \in \mathbb{N}_0}$ and $\beta_1^- = \widetilde{b}_2^{(0)} \geq \widetilde{b}_2^{(1)} \geq \ldots$ is the order statistic of $(\max\{0, b_2^{(j)}\})_{j \in \mathbb{N}_0}$.

Example





$$X_k = \begin{cases} 0.2X_{k-1} + Z_k, & X_{k-1} \leq 0, \\ \pm 0.7X_{k-1} + Z_k, & X_{k-1} > 0. \end{cases}$$

Semi-Heavy Tailed Noise

Long Tailed and Subexponential Distributions

Let F be a distribution function with $F(x) < 1$ for $x \in \mathbb{R}$. Then

- F is in $\mathcal{L}(\gamma)$ with $\gamma \in [0, \infty)$ if

$$\lim_{x \to \infty} \overline{F}(x+y)/\overline{F}(x) = \exp(-\gamma y)$$

holds locally uniformly in $y \in \mathbb{R}$.

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- F is subexponential if $F \in \mathcal{L}(0)$ and

$$\lim_{x \to \infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2.$$

Examples

- $F \in \mathcal{L}(\gamma)$, $\gamma > 0$:
  - $\overline{F}(x) \sim K x^b e^{-\gamma x}$ as $x \to \infty$ where $K > 0$, $b \in \mathbb{R}$
  - GIG (generalized inverse Gaussian) distribution
  - NIG (normal inverse Gaussian) distribution
  - GH (generalized hyperbolic) distribution
  - $CGMY$ distribution
- $F \in \mathrm{MDA}(\Lambda) \cap \mathcal{S}$:
  - $\overline{F}(x) \sim \exp(-x/(\log x)^a)$ as $x \to \infty$ where $a > 0$
  - $\overline{F}(x) \sim K x^b e^{-x^p}$ as $x \to \infty$ where $p \in (0,1)$, $K > 0$ and $b \in \mathbb{R}$
  - lognormal distribution

Tail Behavior

Theorem

Suppose

- $(X_k)_{k\in\mathbb{N}_0}$ be a stationary **TAR(S,q)** process.
- TB and DC hold.
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- $\boldsymbol{Z_1 \in \mathcal{L}(\gamma)}$ with $\boldsymbol{\gamma > 0}$, or that $\boldsymbol{Z_1 \in \mathcal{S} \cap \mathrm{MDA}(\Lambda)}$.

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Then

$$\mathbb{P}(\boldsymbol{X_1} > \boldsymbol{x}) \sim \mathbb{E}\mathrm{e}^{\gamma(\boldsymbol{X_1}-\boldsymbol{Z_1})}\mathbb{P}(\boldsymbol{Z_1} > \boldsymbol{x}) \quad \text{as } \boldsymbol{x} \to \infty.$$

Point Process Convergence

Theorem

Let $a_n > 0$ and $b_n \in \mathbb{R}$ be sequences of constants such that for $x \in \mathbb{R}$

$$\lim_{n \to \infty} n\mathbb{P}(X_0 > a_n x + b_n) = \exp(-x).$$

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Then as $n \to \infty$,

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Then for $x \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}(a_n^{-1}(\max(X_1, \ldots, X_n) - b_n) \leq x) = \exp(-\mathrm{e}^{-x}).$$

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The extremal index is

$$\theta = 1.$$

$\Longrightarrow$ no extremal clusters

Conclusion

- Regularly Varying Noise
  - TAR(S,q) Model: O-regular variation
  - TAR(S,1) Model: regular variation
  - TAR(S,1) Model: extremal clusters and no extremal clusters
- Semi-Heavy Tailed Noise
  - TAR(S,q) Model: semi-heavy tailed
  - TAR(S,q) Model: no extremal clusters