STT 872, 867-868 Fall Preliminary Examination
Wednesday, August 21, 2019
12:30 - 5:30 pm

NOTE: This examination is closed book. Every statement you make must be substantiated. You may do this either by quoting a theorem/result and verifying its applicability or by proving things directly. You may use one part of a problem to solve the other part, even if you are unable to solve the part being used. A complete and clearly written solution of a problem will get a more favorable review than a partial solution.

You must start solution of each problem on the given page. Be sure to put the number assigned to you on the right corner top of every page of your solution.

STT 872, 867-868 Fall Preliminary Examination
Wednesday, August 21, 2019
12:30 - 5:30 pm

1. Let X take on values 1, -1, 0 with probabilities $p, 2p, 1-3p$, respectively.

(a) (2 pts) Describe all random variables $U(X)$ such that $E_pU = 0$ for all $p \in (0, 1/3)$.

(b) (3 pts) Describe all estimands $g(p)$ for which UMVUE exist.

(c) (10 pts) Consider $H : p = 0.25$ vs $K : p < 0.25$. Does a UMP test at level $\alpha \in (0, 1)$ based on X exist? Explain in as much detail as possible.

(d) (5 pts) Let p have the density $\frac{3^{a+1}\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)}p^a(1-3p)^b, a > 0, b > 0, p \in (0, 1/3)$. Find the Bayes estimator of p under the squared loss.

(e) (5 pts) Find conditions under which the estimator in (d) is minimax.

(f) (2 pts) Let $X_1, ..., X_n$ be i.i.d. from the same distribution as X. Find a complete and sufficient statistics for p.

2. Let $X_1, ..., X_n$ be i.i.d. from the density $\frac{\tau}{\sqrt{2\pi x^2}} \exp\{-\frac{\tau^2}{2x^2}\}, \tau > 0$.

(a) (7 pts) Find the MRE estimator of τ^{-2} under the loss function $L(\tau, d) = (d\tau^2 - 1)^2$.

(b) (5 pts) Derive a UMP unbiased test of size $\alpha \in (0, 1)$ for testing $H : \tau = 2$ vs $K : \tau \neq 2$ in as much detail as possible.

(c) (2 pts) Derive the power function of the test in (b).

(d) (2 pts) Construct a UMVUE of $\tau^{-4} + 2\tau^{-2}$.

(e) (2 pts) Propose a consistent estimator of τ.

Hint: Chi-squared distribution with k degrees of freedom has the first moment k, the second moment $k(k+2)$, the third moment $k(k+2)(k+4)$.

3. This problem focuses on linear fixed effects models.

(a) Consider m simple linear regression models (of full rank),

$$y_{ki} = \alpha_k + \beta_k x_{ki} + \epsilon_{ki}, \quad i = 1, 2 \dots, n_k, \quad k = 1, 2, \dots, m,$$

where $\alpha_k \in \mathbb{R}$ is the intercept and $\beta_k \in \mathbb{R}$ is the slope in the kth model. Here, all the ϵ_{ki}'s are independently and identically distributed as $\mathcal{N}(0, \sigma^2)$.

(a1) (2 pts) Write the m models as a single linear regression model in the form

$$Y = X\gamma + \epsilon.$$

Specify X, γ, Y, ϵ. What is the rank of X?

(a2) (3 pts) Suppose we want to test whether the m regression lines are parallel. Give the test statistic and α-level rejection region for the test.

(a3) (3 pts) Suppose the first two regression lines meet at an unknown point (ξ_x, ξ_y). Construct a $100(1-\alpha)\%$ confidence interval for ξ_x.

(a4) (4 pts) [continued from (a3)] Give two methods of constructing simultaneous confidence intervals for $\alpha_1 - \alpha_2, \beta_1 - \beta_2, \xi_x$, with joint coverage probability greater than or equal to $1-\alpha$.

(b) Consider the linear regression model,

$$Y = X\beta + \epsilon, \tag{1}$$

where $Y \in \mathbb{R}^n, X \in \mathbb{R}^{n\times p}, \beta \in \mathbb{R}^p$, and $\epsilon \sim \mathcal{N}(0, \sigma^2 I_n)$. The matrix X is not necessarily of full rank.

(b1) (4 pts) For the Ridge estimator $\hat{\beta}_\lambda = (X'X + \lambda I)^{-1}X'Y$ with $\lambda > 0$, consider the method of generalized cross-validation (GCV) for choosing an appropriate λ. GCV obtains the value of λ that minimizes

$$G(\lambda) = \frac{\|Y - X\hat{\beta}_\lambda\|_2^2}{(1 - \text{trace}(D)/n)^2},$$

where $D = X(X'X + \lambda I)^{-1}X'$. Show that $G(\lambda)$ can be seen as an approximation to the leave-one out cross-validation error $I(\lambda)$ defined as

$$I(\lambda) = \sum_{i=1}^{n} (y_i - x_i'\hat{\beta}_\lambda^{-i})^2,$$

where $\hat{\beta}_\lambda^{-i}$ is the Ridge estimator computed with the ith data removed.

Hint: The Woodbury matrix identity is $(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$, where $A \in \mathbb{R}^{n\times n}, U \in \mathbb{R}^{n\times k}, C \in \mathbb{R}^{k\times k}, V \in \mathbb{R}^{k\times n}$.

(b2) (4 pts) Consider the Lasso estimator given by

$$\hat{\beta} = \arg\min_{\beta} \frac{1}{2}\|Y - X\beta\|_2^2 + \lambda\|\beta\|_1,$$

where $\lambda > 0$ and $\|\cdot\|_1$ is the usual ℓ_1 norm. Let $\mathcal{S} = \{i : \beta_i \neq 0\}, \hat{\mathcal{S}} = \{i : \hat{\beta}_i \neq 0\}$. Suppose there is no error in the model (1) (i.e., $\sigma = 0$), and the following condition holds

$$\|X_{\mathcal{S}^c}' X_{\mathcal{S}} (X_{\mathcal{S}}' X_{\mathcal{S}})^{-1}\|_\infty < 1, \tag{2}$$

where[1] $\mathcal{S}^c$ is the complement of $\mathcal{S}$, and $\|\cdot\|_\infty$ is the matrix norm defined as $\|A\|_\infty = \max_i \sum_j |a_{ij}|$ for a given matrix $A = (a_{ij})$. Use KKT conditions to prove that

$$\hat{\mathcal{S}} \subseteq \mathcal{S},$$

[1] In this problem, for a set $B \subseteq \{1, 2, \ldots, p\}$, $\hat{\beta}_B$ denotes the subvector of $\hat{\beta}$ consisting of the elements indexed by B, and X_B denotes the submatrix of X consisting of the columns indexed by B.

namely, all the variables selected by Lasso are relevant ones.

Hint: It is equivalent to show $\|\hat{\beta}_{S^c}\|_1 = 0$.

4. Consider the following meta-analysis model,

$$y_i = x_i'\beta + b_i + \epsilon_i, \quad i = 1, 2, \ldots, n,$$

where $\beta \in \mathbb{R}^p$ is the fixed effect, $b_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$, $\epsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_i^2)$, and the b_i's and ϵ_i's are independent. Further it is assumed that $\{\sigma_i^2\}_{i=1}^n$ are known and the design matrix $X = (x_1, \ldots, x_n)'$ is of full column rank.

(a) (4 pts) Give the test statistics and α-level rejection region for testing $H_0 : \sigma^2 = 0$ vs $H_1 : \sigma^2 > 0$.

(b) (6 pts) Show that the MLE for β is unbiased.

(c) (10 pts) Obtain the estimator $\hat{\sigma}^2_{mm}$ for σ^2 via the method of moments in the following way:

1. Take the unbiased estimator $\hat{\beta}_{ols} = (\sum_i x_i x_i')^{-1} \sum_i x_i y_i$.
2. Construct the sum of squares of residuals $\text{SSE} = \sum_{i=1}^n (y_i - x_i'\hat{\beta}_{ols})^2$.
3. Solve the equation $\mathbb{E}(\text{SSE}) = \text{SSE}$ for σ^2.

Further show that among all the unbiased estimators that take the form $Y'AY + c$, where $A \in \mathbb{R}^{n\times n}$ is symmetric and c is a constant, $\hat{\sigma}^2_{mm}$ has the smallest norm $\text{trace}(A'A)$, i.e., $\hat{\sigma}^2_{mm}$ is the minimum norm quadratic unbiased estimator (MINQUE).