Prelim January 2021

Preliminary Exam: Probability.

Time: 10:00am - 3:00pm, Friday, January 8, 2021

Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.

The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.
2. Start each problem on a new page.
3. Write only on one side of a page.

Problem 1. Let $\{X_k\}$, $k = 1, 2, \ldots$ be a sequence of random variables. In this problem all convergences are as $k \to \infty$.

a. Prove that $X_k \to 0$ in probability if and only if $X_k \to 0$ in distribution.

b. (i) Prove that if $X_k \to 0$, a.s. then $P(\cup_{m=k}^{\infty}\{|X_m| > \varepsilon\}) \to 0$ for each $\varepsilon > 0$.

   (ii) Can you conclude from (i) that convergence a.s. implies convergence in probability?

c. Assume that $\{X_k\}$ are independent, $P(|X_k| > 1) = \frac{1}{k}$, $k = 1, 2, \ldots$ and

$$P(|X_k| \leq \frac{1}{\log(\log(k))}) \to 1.$$

   Show that $X_k \to 0$ in probability **but** not a.s.

Problem 2. Let $\{X_k\}_{k\geq 1}$ be a sequence of symmetric and independent random variables. The distribution of $\{X_k\}_{k\geq 1}$ is given by

$$X_k = \begin{cases} \pm 1 & with\ probability\ \left(\frac{1}{2}\right)(1 - 2^{-k}) \\ \pm 2^k & with\ probability\ \left(\frac{1}{2}\right)2^{-k} \end{cases}$$

Let $S_n = \sum_{k=1}^{n} X_k$.

a. Prove that $\sum_{k=1}^{\infty} \frac{X_k}{k}$ converge a.s.

b. Is the following statement correct?
   Let $\{Y_k\}_{k\geq 1}$ be a sequence of independent random variables with $E(Y_k) = 0$. If $\sum_{k=1}^{\infty} Y_k$ converge a.s. then $\sum_{k=1}^{\infty} Var(Y_k) < \infty$.

c. Use part a. to conclude that $\frac{S_n}{n} \xrightarrow[n\to\infty]{} 0$, a.s

Problem 3. Let $\{X_k\}_{k\geq 1}$ be a sequence of independent random variables. Let $S_n = \sum_{k=1}^{n} X_k$.

a. Assume that (i) $Var(S_n) \underset{n\to\infty}{\longrightarrow} \infty$, and (ii) there exist a constant $C < \infty$ so that $|X_k| \leq C$ for all $k \geq 1$. Prove that $\frac{S_n - E(S_n)}{\sqrt{Var(S_n)}}$ converge in distribution to $N(0,1)$ as $n \to \infty$.

b. Assume that $E(X_k) = 0, E(X_k^2) = 1$, and that there exist a constant $C < \infty$ so that $E(|X_k|^{2.25}) \leq C$, for all $k \geq 1$. Prove that $\frac{S_n}{\sqrt{n}}$ converge in distribution to $N(0,1)$ as $n \to \infty$.

c. Assume that $Var(S_n) \underset{n\to\infty}{\longrightarrow} a$ with $a < \infty$. Prove that

   (i) $\sum_{k=1}^{\infty} [X_k - E(X_k)]$ converge a.s. and

   (ii) $\frac{S_n - E(S_n)}{\sqrt{Var(S_n)}}$ converge a.s.

Problem 4. Let $\{X, X_k\}$, $k = 1, 2, \ldots$ be a sequence of i.i.d. random variables with

$$P(X = +1) = .25, \quad P(X = -1) = .75.$$

Let $S_0 = 0,\ S_n = \sum_{k=1}^{n} X_k\, , n = 1,2, \ldots$ be an asymmetric, simple random walk and let $F_n = \sigma\{\, X_1\, , \ldots,\ X_n\}, n \geq 1$. F_0 denotes a trivial $\sigma -$ algebra.

a. Prove that
   (i). $\{3^{S_n}, F_n\}, n \geq 0$, and
   (ii). $\{\, S_n + (.5)n, F_n\}, n \geq 0$
   are both martingales.

b. Let $T_x = \inf\{n \geq 0\!:\ S_n = x\}\, , x$ integer.
   Let $a < 0 < b$. It follows from part a(i) that that
   $$P(\, T_a < T_b) = \frac{3^b - 1}{3^b - 3^a}\, , \qquad P(\, T_b < T_a) = \frac{1 - 3^a}{3^b - 3^a}$$
   and you may use those without proof.

   (i) Find $P(T_a < \infty)$ and $P(T_b < \infty)$. Hint: $T_a \underset{a\to-\infty}{\longrightarrow} \infty$, a.s. and $T_b \underset{b\to\infty}{\longrightarrow} \infty$, a.s.

   (ii) Find $E(\max_{n\geq 0} \{S_n\})$. Start by explaining the identity $\{\, \max_{n\geq 0}\{S_n\} \geq b\} = \{T_b < \infty\}$.
   Hint: If X is a random variable with values that are non-negative integers then
   $$E(X) = \sum_{b=1}^{\infty} P(X \geq b).$$

c. Prove that $\{\, (S_n + (.5)n)^2 - (.75)n\, , F_n\}, n \geq 0$ is a martingale.

Problem 5. Let $\{X, X_k\}$, $k = 1, 2, \ldots$ be a sequence of i.i.d. random variables with $X \sim N(0, 9)$. Let $S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$, $n = 1,2, \ldots$. Define: $M_n = \exp(2\, S_n - 18\, n)$, $n \geq 0$ and

$F_n = \sigma\{ X_1 , \ldots , X_n\}, n \geq 1$. F_0 denotes a trivial $\sigma -$ algebra.

a. Prove that $\{M_n, F_n\}$, $n \geq 0$ is a martingale.
b. Prove that M_n converge a.s when $n \to \infty$.
c. Find the limits of the following in a.s. sense:
   (i). $\lim\limits_{n\to\infty} \frac{S_n}{n}$.
   (ii). $\lim\limits_{n\to\infty} M_n$.
d. Is the sequence $\{M_n\}$ uniformly integrable? Hint: what is $E(M_n)$ and what is $E\left(\lim\limits_{n\to\infty} M_n\right)$?

Problem 6. Let $\{B(s):\ s \geq 0\}$ represent standard Brownian motion.

a. Prove that $\lim\limits_{n\to\infty} \frac{B(n)}{n} = 0$ a.s.
b. Prove that $\lim\limits_{t\to\infty} \frac{B(t)}{t} = 0$ a.s.
   [Hint: you may use the fact that $\max\limits_{0\leq s\leq 1} B(s)$ and $|B(1)|$ have the same distribution.]
c. Show that the process $\{W(s):\ s \geq 0\}$ defined by

$$W(s) = \begin{cases} s\, B\left(\frac{1}{s}\right) & if\ s > 0, \\ 0 & if\ s = 0 \end{cases}$$

is also a standard Brownian motion. [Hint: You may use part b.]