Prelim January 2019
Preliminary Exam: Probability.

Time: 10:00am - 3:00pm, Friday, January 18, 2019

Your goal on this exam should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.

2. Start each problem on a new page.

3. Write only on one side of a page.

Problem 1. Let $B = \{B_t,\ t \geq 0\}$ be a standard Brownian motion.

1) a. Let $X_t = tB(t^{-1}),\ t > 0,\ X_0 = 0$. Prove that $\{X_t,\ t \geq 0\}$ is a standard Brownian motion.

   b. Prove: $\lim_{t\to\infty} \frac{B_t}{t} = 0$, a.s. by using part a.

2) Prove that $\left\{e^{B_t - \frac{t}{2}},\ \mathcal{F}_t \quad, t \geq 0\right\}$ is a martingale, where $\{\mathcal{F}_t\}$ denote the canonical filtration of Brownian motion.

3) a. Prove: $\lim_{t\to\infty} e^{B_t - \frac{t}{2}} = 0$, a.s. Hint: Use part b of 1).

   b. Is $\left\{e^{B_t - \frac{t}{2}}\right\},\ t \geq 0$ uniformly integrable? Explain.

Problem 2. Let $\{X, X_k, k = 1, 2, \ldots\}$ be a sequence of independent and identically distributed random variables. Assume that $\frac{X_k}{\sqrt{k}} \xrightarrow[k\to\infty]{} 0$, a.s.

1) a. What is $P(\frac{|X_k|}{\sqrt{k}} \geq 1, \text{infinitely often})$? Explain.

   b. Which of the following is correct: $\sum_{k=1}^{\infty} P(|X_k| > \sqrt{k}) < \infty$
   or $\sum_{k=1}^{\infty} P(|X_k| > \sqrt{k}) = \infty$? Explain.

2) Prove that $E(X^2) < \infty$. Hint: $E(X^2) = \int_{x=0}^{\infty} P(X^2 \geq x)$dx.

3) Assume also that $E(X) = 0$. Denote: $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$ and let $\epsilon > 0$.

   a. Prove that $\sum_{k=1}^{\infty} E(\frac{X_k^2}{k^{1+2\epsilon}}) < \infty$

   b. Prove that $\frac{S_n}{n^{1/2+\epsilon}} \xrightarrow[n\to\infty]{} 0$, a.s.

Problem 3. Let $\{X_k, k = 1, 2, \dots\}$ be a sequence of independent and symmetric random variables. We denote $S_n = \sum_{k=1}^{n} X_k, n \geq 1$. Let $n \geq 1$ and $t > 0$ be fixed.

1) a. What is the relationship between the events $\{\max_{1\leq k\leq n} S_k \geq t\}$ and $\bigcup_{k=1}^{n}\{\max_{1\leq j\leq k-1} S_j < t, S_k \geq t\}$?

   b. Is $\bigcup_{k=1}^{n}\{\max_{1\leq j\leq k-1} S_j < t, S_k \geq t\}$ a union of disjoint events? Explain.

   c. What is the relationship between the events $\{S_n \geq t\}$ and $\bigcup_{k=1}^{n}\{\max_{1\leq j\leq k-1} S_j < t, S_k \geq t, S_n - S_k \geq 0\}$?

2) Prove the following for each $1 \leq k \leq n$.
   a. $S_n - S_k$ is a symmetric random variable. (Hint: What can be said about characteristic function of a symmetric random variable?)

   b. $P(S_n - S_k \geq 0) \geq 1/2$.

   c. The events $\{\max_{1\leq j\leq k-1} S_j < t, S_k \geq t\}$ and $\{S_n - S_k \geq 0\}$ are independent for each $1 \leq k \leq n$.

3) a. Prove Lévy's Inequality: $P(\max_{1\leq k\leq n} S_k \geq t) \leq 2P(S_n \geq t)$, by using the earlier parts of this problem.

   b. Show how to conclude from part a that $P\left(\max_{1\leq k\leq n} |S_k| \geq t\right) \leq 2P(|S_n| \geq t)$.
   Hint: Prove first: $P(\max_{1\leq k\leq n} |S_k| \geq t) \leq P(\max_{1\leq k\leq n} S_k \geq t) + P(\min_{1\leq k\leq n} S_k \leq -t)$.

Problem 4. Let $\{B_t, t \geq 0\}$ be a standard Brownian motion.

1) Let $a > 0$.
   a. Prove that $P(B_1 > a) \leq e^{-a^2/2}$. Hint: Observe that for each $t > 0$ we have $\{B_1 > a\} = \{e^{tB_1} > e^{ta}\}$. Now use the Markov inequality and optimize over $t > 0$.

   b. Prove that $P(|B_1| > a) \leq 2e^{-a^2/2}$.

2) a. Quote the reflection principle for Brownian motion.

   b. Prove that for each $n \geq 1$ and $a > 0$ we have

$$P(\max_{0 \leq t \leq 2^{-n}} |B_t| > a \cdot 2^{-n/2}) \leq 4e^{-a^2/2}.$$

3) Let $\epsilon > 0$ and let $a_n = [(2\ln(2)(1+\epsilon)n]^{1/2}$, $n \geq 1$.

   a. Prove that for each $n \geq 1$ we have

$$P(\max_{0 \leq t \leq 2^{-n}} |B_t| > a_n \cdot 2^{-n/2}) \leq 4 \cdot 2^{-(1+\epsilon)n}.$$

   b. Prove that for each $n \geq 1$ we have

$$P(\max_{0 \leq k \leq 2^n - 1} \max_{k \cdot 2^{-n} \leq t \leq (k+1) \cdot 2^{-n}} |B_t - B_{k \cdot 2^{-n}}| > a_n \cdot 2^{-n/2}) \leq 4 \cdot 2^{-n\epsilon}.$$

   c. Prove that with probability 1 there exist a random $N < \infty$ so that if $n > N$ then

$$\sup_{0 < s < t < 1, |t-s| < 2^{-n}} \{|B_t - B_s|\} < 3 \cdot a_n \cdot 2^{-n/2}$$

Problem 5. Let $\{N, X_k, k = 1, 2, \ldots\}$ be independent random variables. Assume:

(i) $E(X_k) = 0, k = 1, \ldots$,

(ii) $\sup_{k \geq 1}\{E(|X_k|)\} < \infty$

(iii) $E(N) < \infty$,

(iv) N is integer-valued and $N \geq 1$.

Denote: $S_n = \sum_{k=1}^{n} X_k, n \geq 1$.

1) Define the filtration $\mathcal{F}_n = \sigma\{N, X_1, \ldots, X_n\}, n \geq 1$. Prove the following:
a. N is a stopping time with respect to $\{\mathcal{F}_n, n \geq 1\}$.

b. $\{S_n, \mathcal{F}_n, n \geq 1\}$ is a martingale. Hint: why is the independence between N and $\{X_k, k = 1, 2, \ldots\}$ important here?

c. Is $\{S_{n \wedge N}, \mathcal{F}_n, n \geq 1\}$ a martingale as well? Explain or quote a result.

2) Prove that $E(\sum_{k=1}^{N} |X_k|) < \infty$. Hint: Observe that $\sum_{k=1}^{N} |X_k| = \sum_{k=1}^{\infty} |X_k| \cdot 1_{\{N \geq k\}}$.

3) a. Prove that $\{S_{n \wedge N}, n \geq 1\}$ is uniformly integrable. Hint: Find a random variable $Y \geq 0$ so that $E(Y) < \infty$ and $|S_{n \wedge N}| \leq Y, a.s., n \geq 1$.

b. Quote an appropriate optional stopping theorem from which we can conclude that $E(S_N) = 0$.

Problem 6. Let $\{X, X_k, k = 1, 2, \ldots\}$ be a sequence of symmetric, independent and identically distributed random variables. Assume that $P(|X| \geq x) = x^{-2}, x \geq 1$. Let $S_n = \sum_{k=1}^{n} X_k,\ n \geq 1$.

1) a. Calculate $E(|X|)$. If $E(|X|) < \infty$ then calculate also $E(X)$.

b. Calculate $E(X^2)$.

2) Let $Y_{n,k} = X_k \cdot 1_{\{|X_k| < \sqrt{nlog(\log(n))}\}},\ 1 \leq k \leq n,\ n \geq 1$.

a. Find $\lim_{n\to\infty} \sum_{k=1}^{n} P(Y_{n,k} \neq X_k)$.

b. Define $T_n = \sum_{k=1}^{n} Y_{n,k},\ n \geq 1$. Is it true that if we know that $\frac{T_n}{\sqrt{nlog(\mathrm{n})}} \underset{n\to\infty}{\Longrightarrow} N(0,1)$ then we can conclude that $\frac{S_n}{\sqrt{nlog(\mathrm{n})}} \underset{n\to\infty}{\Longrightarrow} N(0,1)$? Explain.

3) a. Prove: $\lim_{n\to\infty} \frac{E(Y_{n,1}^2)}{log(\mathrm{n})} = 1$.

b. Prove: $\frac{T_n}{\sqrt{nlog(\mathrm{n})}} \underset{n\to\infty}{\Longrightarrow} N(0,1)$.