Preliminary Exam: Probability.

Time: 9:30am - 2:30pm, Friday, August 23, 2019.

Your goal should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: **If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!**

Please make sure to apply the following guidelines:

1. On each page you turn in, write your assigned code number. Don't write your name on any page.

2. Start each problem on a new page.

Observe: $n = 1, 2, 3, \ldots$ **in problems 1, 2, 3.**

Problem 1. Let $\{X_n\}$ be a sequence of independent random variables. The distribution of X_n is given by:

$$P(X_n = 2) = P\left(X_n = n^{\beta_n}\right) = a_n, \qquad P(X_n = a_n) = 1 - 2a_n,$$

where $0 < a_n < \frac{1}{3}$ and $\beta_n \in R$.

a. Formulate the two basic Borel- Cantelli lemmas for a sequence of events.

b. Prove: If $\sum_{n=1}^{\infty} a_n < \infty$ then $\sum_{n=1}^{\infty} X_n$ converges a.s.

c. Prove: If $\sum_{n=1}^{\infty} X_n$ converges a.s. then $\sum_{n=1}^{\infty} a_n < \infty$.

Problem 2. Let $\{X_n\}$ be a sequence of random variables. The distribution of X_n is given by $X_n \sim N(0, n)$. We denote the cumulative distribution function(CDF) and the characteristic function of X_n by F_n and φ_n, respectively.

a. Find $\lim_{n\to\infty} F_n(x)$, $x \in R$.

b. Find $\lim_{n\to\infty} \varphi_n(t)$, $t \in R$.

c. (i) Define the concept of a tight sequence of CDFs.
   (ii) Prove that $\{F_n\}$ is not tight.

Problem 3. Let $\{X_n\}$ be a sequence of independent random variables with distribution

$$P(X_n = \pm n^{\alpha}) = \frac{n^{1-2\alpha}}{2}, \qquad P(X_n = 0) = 1 - n^{1-2\alpha},$$

where $\alpha \geq \frac{1}{2}$ is a constant. Denote: $S_n = \sum_{k=1}^{n} X_k$ and $s_n^2 = \mathrm{Var}(S_n)$. It may help to recall that $\sum_{k=1}^{n} k = n(n+1)/2$.

(a) Assume that $\frac{1}{2} \leq \alpha < 1$. Prove:

(i). $\frac{S_n}{s_n}$ converges in distribution to $N(0,1)$ as $n \to \infty$.

(ii). $\frac{S_n}{n}$ converges in distribution to $N\left(0, \frac{1}{2}\right)$ as $n \to \infty$.

(b) Assume that $\alpha > 1$.

(i) Prove that S_n converges a.s. to a random variable as $n \to \infty$.

(ii) Prove that $\frac{S_n}{s_n}$ converges in distribution to 0 as $n \to \infty$.

(c) Assume that $\alpha = 1$. Does Lindberg condition holds for the triangle- array $Y_{n,k} = \frac{X_k}{s_n}$, $1 \leq k \leq n$, $n = 1, 2, \dots$?

Problem 4. In what follows $n = 1, 2, \ldots$ Let $\{W(t):\ 0 \le t \le 1\}$ denote a standard Brownian motion.

Let $\alpha > 1$, $t_n = \alpha^n$ and $f(t) = 2\alpha^2 \mathrm{LL(t)}, t > 0$, where

$\mathrm{LL}(t) = \log(\log(t))$.

a. (i) Prove $P\left(\max_{t_n \le s \le t_{n+1}} \{W(s)\} > \sqrt{t_n f(t_n)}\right) \le P\left(\max_{0 \le s \le t_{n+1}} \left\{\frac{W(s)}{\sqrt{t_{n+1}}}\right\} > \sqrt{\frac{f(t_n)}{\alpha}}\right)$.

(ii) Prove $P\left(\max_{0 \le s \le t_{n+1}} \left\{\frac{W(s)}{\sqrt{t_{n+1}}}\right\} > \sqrt{\frac{f(t_n)}{\alpha}}\right) \le 2P\left(\frac{W(t_{n+1})}{\sqrt{t_{n+1}}} > \sqrt{\frac{f(t_n)}{\alpha}}\right)$.

b. (i) Recall that $P(Z > x) \le e^{-x^2/2}, x > 1$. Prove that there exist $C = C(\alpha)$ so that

$P\left(\max_{t_n \le s \le t_{n+1}} \{W(s)\} > \sqrt{t_n f(t_n)}\right) \le C(\alpha) \cdot n^{-\alpha}$, for n large enough.

(ii) Prove that $P\left(\max_{t_n \le s \le t_{n+1}} \{W(s)\} > \sqrt{t_n f(t_n)}, \text{i.o.}\right) = 0$.

c. (i) Prove that $P\left(\limsup_{t\to\infty} \left\{\frac{W(t)}{\sqrt{2t\mathrm{LL(t)}}}\right\} \le \alpha\right) = 1$.

(ii) Prove that in fact $\limsup_{t\to\infty} \left\{\frac{W(t)}{\sqrt{2t\mathrm{LL(t)}}}\right\} \le 1$, a.s.

Remark: The law of iterated logarithm for Brownian motion says that:

$\limsup_{t\to\infty} \left\{\frac{W(t)}{\sqrt{2t\mathrm{LL(t)}}}\right\} = 1$, a.s.

Problem 5. Let $\{W(t)\colon\ 0 \le t \le 1\}$ denote a standard Brownian motion.

Let $M = \max_{0\le t\le 1}\{W_t\}$ and $T_a = \min\{t \ge 0\colon\ W_t = a\}, a \in R$.

a. (i) Calculate the cumulative distribution function $F_M(x), x \ge 0$

(ii) Calculate the density of M, $f_M(x), x \ge 0$.

Hints: use the reflection principle. Also, $M \ge 0, a.s.$

b. (i) Prove: $P(M > y, W_1 < x) = P(W_1 > 2y - x),\ \ y \ge 0, x \le y$.

(ii) Calculate, $f_{(M,W_1)}(x, y)$, the joint density of (M, W_1). Hint: Use differentiation.

c. (i) Show that $T_M = \min\{t \ge 0\colon\ W_t = M\}$ is not a stopping time w.r.t. the canonical filtration of W. Do it by assuming that T_M is a stopping time and creating a contradiction with the strong markov property. Hint: It follows from part b that $T_M < 1$, a.s.

(ii) The process $\{\ a \to T_a, a \ge 0\}$ is increasing a.s. Prove that this process has a jump up at $a = M$ whose size is at least $1 - T_M$, i.e.

$$\lim_{\substack{h\to 0\\ h>0}}\{T_{M+h}\} - T_M \ge 1 - T_M, \text{a.s.}$$

Problem 6. Let $\{X_n, \mathcal{F}_n\}_{n\geq 0}$ be a sequence of L^2 martingales with $X_0 = 0$. Denote the martingale-difference sequence by $D_n = X_n - X_{n-1}$, $n \geq 1$. Denote $A_n = X_n^2 - \sum_{k=1}^{n} D_k^2$ and $B_n = X_n^2 - \sum_{k=1}^{n} E_{\mathcal{F}_{k-1}}(D_k^2)$, $n \geq 1$.

a. Prove $\{A_n, \mathcal{F}_n\}_{n\geq 1}$ is a martingale.

b. (i) Prove that $\{B_n - A_n, \mathcal{F}_n\}_{n\geq 1}$ is a martingale.

(ii) Show how you can conclude from part a and b(i) that $\{B_n, \mathcal{F}_n\}_{n\geq 1}$ is a martingale as well.

c. (i) Prove that $\{X_n^2, \mathcal{F}_n\}_{n\geq 0}$ is a submartingale.

(ii) Find Doob's decomposition of $\{X_n^2, \mathcal{F}_n\}_{n\geq 0}$.