Preliminary Exam: Probability.

Time: 10:00am - 3:00pm, Friday, August 24, 2018.

Your goal should be to demonstrate mastery of probability theory and maturity of thought. Your arguments should be clear, careful and complete.
The exam consists of six main problems, each with several steps designed to help you in the overall solution.

Important: **If you cannot solve a certain part of a problem, you still may use its conclusion in a later part!**

<u>Please make sure to apply the following guidelines:</u>

1. On each page you turn in, write your assigned code number. Don't write your name on any page.

2. Start each problem on a new page.

Prelim in Probability August 2018

Problem 1.

Let $Z \sim N(0,1)$ and denote by $f_Z(x),\ -\infty < x < \infty$ the density of Z.

a. Prove for each $a < b$

$$E(Z; a < Z < b) = f_Z(a) - f_Z(b).$$

b. Let $f, f' : \mathcal{R} \to \mathcal{R}$ be two continuous functions where f' is the derivative of f. Assume that there exists a positive integer k and a constant $0 \le C < \infty$ so that both $\limsup_{|x|\to\infty} \frac{|f(x)|}{|x|^k} \le C$ and $\limsup_{|x|\to\infty} \frac{|f'(x)|}{|x|^k} \le C$ hold (In words: f and f' have at most polynomial growth.)

(i) Prove that $E(|f'(Z)|) < \infty$ and $E(|Zf(Z)|) < \infty$

(ii) Prove that $E\big(f'(Z)\big) = E\big(Zf(Z)\big)$. Hint: Use integration by parts.

c. Show how we can get the following 2 identities from part b(ii).

(i) $E(Z^{n+1}) = nE(Z^{n-1}),\ n \ge 1.$

(ii) Let f, f' be as in part b and let g, g' be another pair of continuous functions, g' is the derivative of g and both have at most polynomial growth. Then

$$E(f'(Z)g(Z)) = E(Zf(Z)g(Z)) - E(f(Z)g'(Z))$$

Problem 2.

Let $Z, Z_1, Z_2, \ldots$ be iid sequence of random variables where $Z \sim N(0,1)$ and we denote by $f_Z(x)$, $-\infty < x < \infty$ the density of Z. Also, let $M_n = \max_{1 \le k \le n} \{Z_k\}$.

a. Recall the inequality $\left(\frac{1}{x} - \frac{1}{x^3}\right) f_Z(x) \le P(Z > x) \le \left(\frac{1}{x}\right) f_Z(x),\ x > 0.$

(i). Prove that $P(Z > x) \sim \left(\frac{1}{x}\right) f_Z(x)$, namely $\frac{P(Z>x)}{\left(\frac{1}{x}\right) f_Z(x)} \underset{x\to\infty}{\longrightarrow} 1.$

(ii). Prove that for each $y \in \mathcal{R}$ we have $\frac{P\left(Z>x+\frac{y}{x}\right)}{P(Z>x)} \underset{x\to\infty}{\longrightarrow} e^{-y}$. Hint: use (i).

b. Let a_n satisfy $P(Z > a_n) = \frac{1}{n},\ n \ge 1$. Let $q_n(y) = P\left(Z > a_n + \frac{y}{a_n}\right), n \ge 1.$

(i) Prove that $(1 - q_n(y))^n \underset{n\to\infty}{\longrightarrow} e^{-e^{-y}},\ y \in \mathcal{R}.$

Hint: Show first (by using part a) that $nq_n(y) \underset{n\to\infty}{\longrightarrow} e^{-y}.$

(ii) Prove that $a_n(M_n - a_n) \underset{n\to\infty}{\Longrightarrow}$ Y where $F_Y(y) = e^{-e^{-y}}, y \in \mathcal{R}$. Hint: Show that

$P(a_n(M_n - a_n) \le \mathrm{y}) \underset{n\to\infty}{\longrightarrow} e^{-e^{-y}},\ y \in \mathcal{R}.$

c. Prove that $M_n - a_n \underset{n\to\infty}{\longrightarrow} 0$ in probability and conclude that $M_n/a_n \underset{n\to\infty}{\longrightarrow} 1$ in probability.

Problem 3.

Let $X, X_1, X_2, \ldots$ be iid sequence of random variables where $\sim Bernoulli(\frac{1}{2})$. Let

$S_n = \sum_{k=1}^{n} \frac{X_k}{2^k},\ n \geq 1.$

a. Prove the following

(i) S_n converges a.s. as $n \to \infty$. Denote the limit random variable by S_∞.

(ii) $E(S_n) \underset{n\to\infty}{\longrightarrow} E(S_\infty)$ and $V(S_n) \underset{n\to\infty}{\longrightarrow} V(S_\infty)$. Calculate $E(S_\infty)$ and $V(S_\infty)$.

b. Prove that $\prod_{k=1}^{n} \frac{1+e^{i(\frac{t}{2^k})}}{2}$ converges as $n \to \infty$ for each $t \in \mathcal{R}$.

c. It is known that $\prod_{k=1}^{\infty} \frac{1+e^{i(\frac{t}{2^k})}}{2} = \frac{e^{it}-1}{it}$, $t \in \mathcal{R}$. Use this identity to find the distribution of S_∞.

Problem 4.

Let $(\Omega, \mathcal{F}, P)$ be a probability space that supports a standard Brownian motion $\{B(t), 0 \leq t \leq 1\}$. Let $([0,1], \mathcal{B}, \lambda)$ be a measure space with $\mathcal{B}$ the borel σ-algebra and λ the Lebesgue measure. Let

$(S = \Omega \times [0,1], \mathcal{G} = \mathcal{F} \times \mathcal{B}, \mu = P \times \lambda)$ denote the product of the two spaces. It is known that the function $h: S \to \mathcal{R}$ defined by $h(\omega, t) = B_t(\omega)$ is a measurable function on $(S, \mathcal{G})$, so you can use this fact without proof. (It can be easily proved by using the sample continuity of Brownian motion.)

a. Prove that $h^k, k \geq 0$ is integrable on S, namely prove that $\int_S |h^k| d\mu < \infty$.

b. Let $X(\omega) = \int_{t=0}^{1} B_t(\omega) dt, \ \ \omega \in \Omega$.

(i) Prove (or quote a theorem) that X is a random variable, namely it is measurable.

(ii) Let $s \in [0,1]$. Calculate $E(B_s X)$. Justify your steps.

c. Calculate $E(X^2)$. Hint: $\left(\int_{t=0}^{1} B_t dt\right)^2 = \left(\int_{t=0}^{1} B_s ds\right)\left(\int_{t=0}^{1} B_t dt\right)$.

Problem 5.

Let $\{B_t,\ t \geq 0\}$ be standard Brownian motion(SBM) and let $X, X_1, X_2, \ldots$ be i.i.d. random variables with $E(X) = 0$ and $E(X^2) < \infty$. We denote: $S_n = \sum_{k=1}^{n} X_k, n \geq 1$.
Assume $0 < T_1 < T_2 < \cdots$ be a sequence of stopping times(with respect to the canonical filtration of SBM) so that

(1) $S_n = B_{T_n}$ in distribution , $n \geq 1$,

(2) $\{T_1, T_2 - T_1, T_3 - T_2 \ldots\}$ are i.i.d sequence, and

(3) $E(T_1) = E(X^2)$.

Remark: A. Skorohod proved that a sequence of stopping times with those properties always exist.

a. Prove that $\frac{T_n}{n} \underset{n\to\infty}{\longrightarrow} E(X^2)$, a.s.

b. For each $n \geq 1$ let $W_n(t) = \frac{B(nt)}{\sqrt{n}},\ t \geq 0$. Prove first that W_n is a SBM and then prove that $W_n\left(\frac{T_n}{n}\right) = \frac{S_n}{\sqrt{n}}$ in distribution.

c Let $\{H, Y_n, Z_n, n \geq 1\}$ be a family of random variables. Prove that if

$Y_n - Z_n \underset{n\to\infty}{\longrightarrow} 0$, in probability, and $Z_n \underset{n\to\infty}{\Longrightarrow}$ H then $Y_n \underset{n\to\infty}{\Longrightarrow}$ H as well.

d. Prove that $\frac{S_n}{\sqrt{n}} \underset{n\to\infty}{\Longrightarrow}$N(0, $E(X_1^2)$) by using parts a, b and c

Hint: You need to prove that $W_n\left(\frac{T_n}{n}\right) - W_n\left(E(X_1^2)\right) \underset{n\to\infty}{\longrightarrow} 0$, in probability.

Problem 6.

Let $\{X_n\}, \{Y_n\}, \{Z_n\}$ $n \geq 0$ be 3 sequences of random variables that are **integrable**, non-negative and adapted to the filtration $\{ \mathcal{F}_n\}, n \geq 0$. We assume that $\sum_{k=1}^{\infty} Y_n < \infty, a.s.$

a. Prove that $\prod_{k=1}^{n}(1 + Y_k)^2$, $n \geq 1$ is a non-decreasing sequence of random variables that convergences to $\prod_{k=1}^{\infty}(1 + Y_k)^2 < \infty$, where all statements are in a.s. sense.

From now we assume: $E_{\mathcal{F}_n}(X_{n+1}) \leq X_n(1 + Y_n)^2$ and $E_{\mathcal{F}_n}(Z_{n+1}) \leq Z_n(1 + Y_n)^2$, $n \geq 0, a.s.$

b.(i) Prove that $\left\{\frac{X_n}{\prod_{k=1}^{n-1}(1+Y_k)^2}, \mathcal{F}_n\right\}, n \geq 1$, is a non-negative SUPERMG sequence.

(ii) Prove that $\left\{\frac{\min(X_n, Z_n)}{\prod_{k=1}^{n-1}(1+Y_k)^2}, \mathcal{F}_n\right\}, n \geq 1$, is also a non-negative SUPERMG sequence.

Hint for (ii): You need to show that if $\{H_n, \mathcal{F}_n\}$ and $\{J_n, \mathcal{F}_n\}$ are both SUPERMG then so is $\{\min(H_n, J_n), \mathcal{F}_n\}$.

c. Prove that X_n and $\min(X_n, Z_n)$ both converges a.s. to a finite limit.